Università di Pisa

Master's Thesis

Relations between Hamiltonian Cellular Automata and Quantum Mechanics of composite systems

Author:
Alessio Andreoni

Supervisor:
Prof. Hans-Thomas Elze

Anno Accademico 2014/2015

UNIVERSITÀ DI PISA

Abstract

Facoltà di Fisica
Dipartimento di Fisica "Enrico Fermi"

Relations between Hamiltonian Cellular Automata and Quantum Mechanics of composite systems

by Alessio ANDREONI

In this Thesis we will explore the features of the Hamiltonian Cellular Automaton (HCA) defined in [1]. In particular, we will concentrate on apparent similarities with quantum mechanics. We will see that a HCA follows updating equations that represent a discretized analogue of Schrödinger's equation and conservation laws similar to those of quantum mechanics. Once we have introduced an HCA and studied its main properties, we will consider the composition of two of them. It is possible to do that in various different ways: first, following the composition rule of classical systems and second, trying to mimick as close as possible the quantum mechanical procedure.

Acknowledgements

I would like to thank my family and all my professors, in particular my supervisor Prof. Hans-Thomas Elze, and Professors Giovanni Morchio, Franco Strocchi and Mihail Mintchev for helpful conversations.

Contents

List of Figures

Chapter 1

Introduction

Cellular Automata (CA) are an idealization of a physical system in which space and time are discrete and the physical quantities take only a finite set of values. The concept of Cellular Automata dates back to the late 1940s and is due to John von Neumann. Von Neumann was thinking of imitating the behaviour of a human brain in order to build a machine able to solve complex problems. However, his motivation was more ambitious than just to achieve a performance increase of the computer of that time. He thought that a machine with a complexity comparing to the brain should also contain self-control and self-repair mechanisms [2].

A Cellular Automaton is, in general, a set of cells distributed on a lattice in a D-dimensional space. Each cell is characterized by an internal state, which typically corresponds to a finite number of information bits. This system of cells evolves, in discrete time steps, following a simple recipe to compute their new internal state. This recipe can be either deterministic or probabilistic.

The rule determining the evolution of each cell is a function of the state of the cell itself and of its nearest neighbour cells. All the cells evolve simultaneously.

A class of Cellular Automata has the so called property of universal computation. This means that there exists an initial configuration of the Cellular Automaton which leads to the solution of any (computer) algorithm. Even if this property is more of theoretical than of practical interest.

Cellular Automata are used in many different fields, such as equilibrium and non-equilibrium statistical physics, application-oriented problems, biology, sociology, chemistry and others. Some examples may illustrate this versatility.

We can use a Cellular Automaton to mimic gas diffusion. The CA doing that is quite simple: we have a two-dimensional grid of cells, each cell can have two different values attached to it, say 1 or 0. If the value is 1, the cell is occupied by a gas particle, if the value is 0, it is empty. Then the updating rule is applied based on Margolus-neighbourhood blocks that consist in dividing the grid into 2×2 blocks; considering this grid and the one shifted by one cell along the diagonal, they are called the odd and even grid, respectively. Then, taking alternatively the odd and even partitioning, the rule for each Margolus-neighbourhood block consists in rotating it clockwise or counter-clockwise by $\pi/2$, depending on the outcome of a coin toss. In this way, if we start with a cluster of particles in a restricted area, we will see them diffuse.

There are also Cellular Automata which obey the Navier-Stokes equation, either approximately or exactly, as the HPP-GAS rule which has the following formulation. Consider an orthogonal lattice consisting of sites connected by north, south, west, and east links. There are four kinds of particles, one for each direction, and a site can be occupied by at most one particle of each kind. The updating consists in a two-step cycle. In the first step, each particle moves along a link from its current site to the adjacent site in its direction, in the second step, particles are shuffled in the following way: if there are at that site exactly two particles which have come in from opposite directions, say north and south, they are replaced by a west-east pair; otherwise nothing changes, and correspospondingly for the other pair of directions. This Cellular Automaton follows the Navier-Stokes equation only in an approximate way, in fact the viscosity is anisotropic.

More details on these examples can be found in [3].

In biology Cellular Automata are used to model ecosystems [4], the behaviour of prey-predator systems [5], for the recontruction of DNA sequences [6], and the growth of tumors [7], etc.

In social science they are used to model the dynamics of large groups of individuals [8], especially in the study of social dilemmas [9].

In chemistry they are used to model chemical turbulence [10].

Moreover, recently, physicists such as G. 't Hooft have been proposing Cellular Automata as possible hidden variables theory underlying quantum mechanics ([11], [12], [13]). In his proposals mostly no specific models are introduced (except, e.g., one in string theory), but the author often talks about the general features that possible models must have. We mention that some of the conjectures made by the author could prove wrong as pointed out in [14]. This leads, in particular, to the problem that an underlying deterministic theory for quantum mechanics may require superdeterminism and/or non-locality, in order to explain Bell's inequalities.

One of the proposals of this Thesis is to explore a class of Cellular Automata - that we will call Hamiltonian Cellular Automata (HCA) - in order to understand if they can show some of the features of quantum mechanical systems.

In Chapter (2), we will introduce the concept of Cellular Automata, then we will generalize it to include the case in which there is no spatial lattice. After that we will talk about the general characteristics of the HCAs we are interested in, presenting their action and deriving their updating equations and conservation laws.

In Chapter (3), we will study more accurately the space of states and the observables of the HCAs, trying to build a structure that could resemble the corresponding one of quantum mechanics. In this chapter, we will introduce the concept of the algebra of observables and of C*-algebras and we will rewrite the updating equations, the conservation laws, and the action using the states we built. This study will be useful in the next chapter to describe composite HCAs.

An important point, which has found little attention in the literature, so far, if one pursues the study of parallels between HCAs and quantum mechanical systems, is the behaviour of composite systems. It is of crucial importance for most applications of quantum mechanics, since it touches upon its most distinctive feature, namely entanglement.

Therefore, in Chapter (4), we will try to combine two HCAs in two different ways: using the direct sum of their space of states, as is done in classical mechanics, and their tensor product, as in quantum mechanics. Correspondingly we will rewrite the action for composite systems, their updating equations, and the conservation laws, based on our new construction of composite systems here.

In the last Chapter (5), we will show the results of some numerical studies to better understand the differences between the HCAs and quantum mechanics, both, in the single system case and in the composite one. We will find here also some restriction on the eigenvalues of the Hamiltonians that can possibly be used to evolve an HCA.

In Chapter (6), we present our conclusions and a perspective on this work and future extensions.

Chapter 2

Cellular Automata

2.1 Formal definition

We present a formal definition of a Cellular Automaton following what has been done in [2] and then we generalize this idea. A Cellular Automaton requires:

(i) a regular lattice of cells covering a portion of a D dimensional space;

(ii) a set $\psi(\vec{r},t) = \{\psi_1(\vec{r},t), \psi_2(\vec{r},t), ..., \psi_l(\vec{r},t)\}$ of discrete variables attached to each site $\vec{r}$ of the lattice and giving the local state of each cell at the time $t = 0, 1, ...$;

(iii) a rule $R = \{R_1, R_2, ..., R_l\}$ which specifies the time evolution of the state $\psi(\vec{r},t)$:

$$\psi_j(\vec{r}, t+1) = R_j \left(\psi(\vec{r},t), \psi(\vec{r}+\vec{\delta_1},t), ..., \psi(\vec{r}+\vec{\delta_q},t) \right) \ . \tag{2.1}$$

In this definition, the new state at time $t+1$ is only a function of the previous state at time t. It is sometimes necessary to have a longer memory and introduce a dependence on the states at times $t-1, t-2, ..., t-k$. Such a situation, however, is already included in the definition given above, if one keeps a copy of the previous state in the current state. Extra variables can be defined for this purpose.

For example, the one-dimensional second order rule:

$$\psi_1(r, t+1) = R\left(\psi_1(r-1,t), \psi_1(r,t-1)\right) \ , \tag{2.2}$$

can be expressed as a first order rule introducing a new state $\psi_2(\vec{r},t)$ as follows:

$$\psi_1(r, t+1) = R\left(\psi_1(r-1, t), \psi_2(r, t)\right) , \tag{2.3}$$

$$\psi_2(r, t+1) = \psi_1(r, t) . \tag{2.4}$$

For our purposes, we will not use the given definition of a Cellular Automaton but a more general one.

Our Generalized Cellular Automaton (GCA) will require:

(i) a denumerable set of variables $\psi^\alpha(t)$, where α is a multi-index which denotes different degrees of freedom, giving the state of each cell at the time $t = 0, 1, \ldots$;

(ii) a denumerable set of rules R^α which specifies the time evolution of the state $\psi^\alpha(t)$ in the following way:

$$\psi_j^\alpha(t+1) = R^\alpha\left(\{\psi^\beta(t)\}\right) . \tag{2.5}$$

We recover the previous definition for $\alpha = \{\vec{r}, i\}$, with $\vec{r}$ denoting the position of the cell on a D dimensional lattice, $R_i^{\vec{r}}$ does not depend on $\vec{r}$, and with $i = 1, \ldots, l$

According to their above definitions CA and GCA are deterministic (unless we introduce rules that are explicitly random, as for the gas diffusion Cellular Automaton mentioned in the previous chapter). The rules are some well-defined functions, and a given initial configuration will always evolve in the same way.

What we want to do next is to introduce a GCA called Hamiltonian Cellular Automaton which has a discrete time evolution equation which will turn out to be related to the Schrödinger time evolution equation of quantum mechanics, similarly as in [1].

2.2 Hamiltonian Cellular Automaton (HCA)

Consider a Generalized Cellular Automaton (we will call it Hamiltonian Cellular Automaton or HCA) with a denumerable set of degrees of freedom and represent its state by "coordinates" x_n^α, τ_n and "conjugated momenta" p_n^α, π_n, where α is an integer multi-index and $n \in \mathbb{Z}$ denotes different states. Note that we introduced time (τ_m) as a dynamical variable. In the next section, we will consider the same HCA without the variables τ_n and π_n because in what follows we will always consider only trivial evolution for the time variable, so it will be easier to eliminate it. The only reason to introduce it, is to put on an equal footing time and space, as has been suggested in [15].

We will consider the variables x_n^α and p_n^α real ones. This because in Chapter (3), we will need to build a Hilbert vector with them, and in order to do that we need real variables.

The x_n^α and p_n^α might be higher-dimensional vectors, while τ_n and π_n are assumed one-dimensional. Finite differences for all dynamical variables are defined by:

$$\Delta f_n := f_n - f_{n-1} \ . \tag{2.6}$$

We also define (using the summation convention for repeated greek indices, $r^\alpha s^\alpha = \sum_\alpha r^\alpha s^\alpha$):

$$A_n := \Delta\tau_n (H_n + H_{n-1}) + a_n \ , \tag{2.7}$$

$$H_n := \frac{1}{2} S_{\alpha\beta}(p_n^\alpha p_n^\beta + x_n^\alpha x_n^\beta) + A_{\alpha\beta} p_n^\alpha x_n^\beta + R_n \ , \tag{2.8}$$

$$a_n := c_n \pi_n \ , \tag{2.9}$$

where c_n are constants, $\hat{S} = \{S_{\alpha\beta}\}$ is a symmetric matrix, $\hat{A} = \{A_{\alpha\beta}\}$ is an antisymmetric matrix, and R_n stands for higher than second power in x_n^α or p_n^α. The choice of a_n influences the behaviour of the variable τ_n.

We define the HCA action:

$$S := \sum_n [(p_n^\alpha + p_{n-1}^\alpha)\Delta x_n^\alpha + (\pi_n + \pi_{n-1})\Delta\tau_n - A_n] \ . \tag{2.10}$$

Then the evolution of the HCA is determined by the following postulate.

Postulate A. The HCA follows the discrete updating rules (equations of motion) which are determined by the action principle $\delta S = 0$, referring to variations of all dynamical variables defined by:

$$\delta g(f_n) := \frac{1}{2}[g(f_n + \delta f_n) - g(f_n - \delta f_n)] \ , \tag{2.11}$$

where f_n stands for one of the variables on which polynomial g may depend.

In eq.(2.11), we can consider arbitrary variations or just infinitesimal ones. If we consider arbitrary variations we have that the variations of constant, linear, or quadratic terms yield results that are similar to the continuum case, while the variations of higher order terms are different. While, if we consider just infinitesimal variations, also the variation of higher orders terms would be similar to the continuum case. In the first case, we will refer to **Postulate A** as **Strong Postulate A**.

For arbitrary δf_n the remainder of higher powers in eq.(2.8) has to vanish for consistency, $R_n = 0$. In fact its variation generates additional equations of motion such that the number of these exceeds the number of variables. How to work out the equations of motion from the action principle is shown in Appendix A. Here we only give the resulting updating equations.

Let us introduce the notation $\dot{O}_n := O_{n+1} - O_{n-1}$, then the GCA equations of motion are:

$$\dot{x}_n^\alpha = \dot{\tau}_n(S_{\alpha\beta}p_n^\beta + A_{\alpha\beta}x_n^\beta) \ , \tag{2.12}$$

$$\dot{p}_n^\alpha = -\dot{\tau}_n(S_{\alpha\beta}x_n^\beta - A_{\alpha\beta}p_n^\beta) \ , \tag{2.13}$$

$$\dot{\tau}_n = c_n \ , \tag{2.14}$$

$$\dot{\pi}_n = \dot{H}_n \ , \tag{2.15}$$

which are discrete analogues of Hamilton's equations (therefore, from now on, we will call our GCA an Hamiltonian Cellular Automaton (HCA)). The discrete automaton time n is reflected by the finite difference equations. - Equations (2.12)-(2.15) are time reversal invariant; we can obtain the state at time $n+1$ from the knowledge of the earlier states at n and $n-1$ or we can as well obtain the state at time $n-1$ from the later ones at n and $n+1$.

2.3 Solution for the updating equations

Now we want to write down a formal solution for the variables at time n.

To do that we introduce the self-adjoint matrix $\hat{H} := \hat{S} + i\hat{A}$ and the variables $\psi_n^\alpha := (1/\sqrt{2})(x_n^\alpha + ip_n^\alpha)$, so that we can rewrite the action S as:

$$\tag{2.16}$$

where we have:

$$A'_n := \Delta\tau_n(H'_n + H'_{n-1}) + a_n \ , \tag{2.17}$$

$$H'_n := H_{\alpha\beta}\psi_n^{*\alpha}\psi_n^\beta \ , \tag{2.18}$$

a_n is the same of equation (2.9).

In the new variables, equations (2.12) and (2.13) become:

$$\dot{\psi}_n^\alpha = -i\dot{\tau}_n H_{\alpha\beta}\psi_n^\beta \ , \tag{2.19}$$

$$\dot{\psi}_n^{*\alpha} = i\dot{\tau}_n H_{\alpha\beta}^*\psi_n^{*\beta} \ . \tag{2.20}$$

Equations (2.19), (2.20) come from the variation of the action S in eq.(2.16) w.r.t. the new variables $\psi_n^{*\alpha}$ and ψ_n^α. So we recover a discrete analogue of the Schrödinger equation, and its adjoint, if we consider ψ_n^α as the amplitude of the "α-component" of a "state vector" $|\psi\rangle$.

As as been shown in [16], we can write the solutions of eq.(2.20) using the Chebyshev polynomials $U_i(x)$ of the second kind (x is the argument of the polynomial that in our case is the matrix $-c\hat{H}$). They are defined as follows:

$$\begin{gathered} U_0(x) = \mathbb{I} \ , \\ U_1(x) = 2x \ , \\ \vdots \\ U_{n+1} = 2xU_n(x) - U_{n-1}(x) \ . \end{gathered} \tag{2.21}$$

Given the two initial conditions ψ_0^α and ψ_1^α, one obtains:

$$\psi_n^\alpha = -i^n \left[\left(U_{n-2}\left(-\frac{\dot{\tau}}{2}\hat{H}\right)\right)_{\alpha\beta} \psi_0^\beta + i \left(U_{n-1}\left(-\frac{\dot{\tau}}{2}\hat{H}\right)\right)_{\alpha\beta} \psi_1^\beta \right] . \tag{2.22}$$

Moreover, if instead of starting with ψ_0^α and ψ_1^α we would have started with ψ_k^α and ψ_{k+1}^α, we would find:

$$\psi_n^\alpha = -i^{n-k} \left[\left(U_{n-k-2}\left(-\frac{\dot{\tau}}{2}\hat{H}\right)\right)_{\alpha\beta} \psi_k^\beta + i \left(U_{n-k-1}\left(-\frac{\dot{\tau}}{2}\hat{H}\right)\right)_{\alpha\beta} \psi_{k+1}^\beta \right] , \tag{2.23}$$

that is a sort of composition rule for the time translation operators. For a derivation of the solution (2.22) and of the eq.(2.23), see [16], Chapter 1.2, and for a general insight on the discretized Schrödinger equation see [17]

2.4 Conservation laws

We can easily see that equations (2.19), (2.20), and so equations (2.12), (2.13), imply the following theorem on conservation laws.

Theorem A. For any matrix $\hat{G}$ that commutes with $\hat{H}$, $[\hat{G}, \hat{H}] = 0$, there is a discrete conservation law:

$$\psi_n^{*\alpha} G_{\alpha\beta} \dot{\psi}_n^\beta + \dot{\psi}_n^{*\alpha} G_{\alpha\beta} \psi_n^\beta = 0 . \tag{2.24}$$

Proof. Sobstituting expressions (2.19) (2.20) for $\dot{\psi}_n^\beta$ and $\dot{\psi}_n^{*\alpha}$ we obtain:

$$\begin{aligned} \psi_n^{*\alpha} G_{\alpha\beta} \dot{\psi}_n^\beta + \dot{\psi}_n^{*\alpha} G_{\alpha\beta} \psi_n^\beta &= -i\dot{\tau}_n(\psi_n^{*\alpha} G_{\alpha\beta} H_{\beta\gamma} \psi_n^\gamma - \psi_n^{*\gamma} H^*_{\alpha\gamma} G_{\alpha\beta} \psi_n^\beta) \\ &= -i\dot{\tau}_n(\psi_n^{*\alpha} G_{\alpha\beta} H_{\beta\gamma} \psi_n^\gamma - \psi_n^{*\alpha} H_{\alpha\beta} G_{\beta\gamma} \psi_n^\gamma) \\ &= -i\dot{\tau}_n(\psi_n^{*\alpha} [G, H]_{\alpha\beta} \psi_n^\beta) = 0 . \qquad \square \end{aligned} \tag{2.25}$$

In particular, if we take $\hat{G} = \hat{\mathbb{I}}$ we get:

$$\psi_n^{*\alpha} \dot{\psi}_n^\alpha + \dot{\psi}_n^{*\alpha} \psi_n^\alpha = 0 . \tag{2.26}$$

Equation (2.26) is very similar to the probability current conservation law of quantum mechanics. Note that while in quantum mechanics we can integrate it, to get the norm conservation law, here this is not possible because of the definition of the discrete time derivative, recall $\dot{O}_n := O_{n+1} - O_{n-1}$. Indeed the quantity:

$$\psi_n^{*\alpha}\psi_n^{\alpha} \ , \tag{2.27}$$

is not conserved.

For $\hat{G} = \hat{H}$ we have an "energy conservation" law:

$$\psi_n^{*\alpha}H_{\alpha\beta}\dot{\psi}_n^{\beta} + \dot{\psi}_n^{*\alpha}H_{\alpha\beta}\psi_n^{\beta} = 0 \ . \tag{2.28}$$

We can rewrite equation (2.24), replacing $\dot{\psi}$ with $\psi_{n+1} - \psi_{n-1}$ and introducing the symmetric two-time function:

$$2C_{\hat{G}}(m,n) = \psi_m^{*\alpha}G_{\alpha\beta}\psi_n^{\beta} + c.c. \ , \tag{2.29}$$

where $X + c.c. := X + X^*$; obtaining the conservation law:

$$C_{\hat{G}}(n-1,n) = C_{\hat{G}}(n,n+1) \ . \tag{2.30}$$

Equation (2.30) tells us that $C_{\hat{G}}(n-1,n)$ does not depend on n, so it is the conserved quantity.

Of particular interest is the conserved quantity $C_{\hat{\mathbb{I}}}(n,n+1)$. Later we will see, that in order to recognize some of the quantum mechanical structure in an HCA, we may introduce a time scale l and consider the limit $l \to 0$. Thus the conserved quantity $C_{\hat{\mathbb{I}}}(n,n+1)$ will become the norm of the state; in this way the norm conservation law of quantum mechanics can be recovered.

2.5 Eliminating τ_m and π_m as dynamical variables

Now we want to eliminate the dynamical time variables τ_n and π_n. We keep eq.(2.6) as the definition of finite differences between variables. Next, we define the quantities:

$$A_n := 2cH_n \ , \tag{2.31}$$

$$H_n := \frac{1}{2} S_{\alpha\beta}(p_n^\alpha p_n^\beta + x_n^\alpha x_n^\beta) + A_{\alpha\beta} p_n^\alpha x_n^\beta \ , \tag{2.32}$$

where c is a constant, $\hat{S} = \{S_{\alpha\beta}\}$ is a symmetric matrix, $\hat{A} = \{A_{\alpha\beta}\}$ is an antisymmetric matrix. Then, we use as a definition for the action S:

$$S := \sum_n [(p_n^\alpha + p_{n-1}^\alpha)\Delta x_n^\alpha - A_n] \ . \tag{2.33}$$

Defining as before $\dot{O}_n = O_{n+1} - O_{n-1}$ and applying the **Strong Postulate A**, we derive the updating equations for x_n^α and p_n^α:

$$\dot{x}_n^\alpha = 2c(S_{\alpha\beta} p_n^\beta + A_{\alpha\beta} x_n^\beta) \ , \tag{2.34}$$

$$\dot{p}_n^\alpha = -2c(S_{\alpha\beta} x_n^\beta - A_{\alpha\beta} p_n^\beta) \ . \tag{2.35}$$

These are the same as eqs. (2.12) and (2.13) with $2c$ in place of $\dot{\tau}_n$. Because of this and the fact that we can apply the change of variables we applied in the previous case also here, and because eqs. (2.14) and (2.15) do not influence directly the conservation laws of the HCA, also in this case we have the validity of **Theorem A**.

The corresponding updating equation in term of $\psi_n^\alpha = x_n^\alpha + ip_n^\alpha$ is:

$$\dot{\psi}_n^\alpha = -icH_{\alpha\beta}\psi_n^\beta \ , \tag{2.36}$$

From now on, we will work exclusively with this definition of our HCA (without time as a dynamical variable).

2.6 Discrete Euler-Lagrange equations

In Sections 2.2 and 2.5, we have written the action for our HCA in terms of the complex variable ψ_n^α. Now we want to show that the integrand of eq.(2.31) can be considered a Lagrangian and that we can rewrite the updating equations (2.34) and (2.35) as discrete analogue of Euler-Lagrange equations. To do that we start rewriting the action in (2.31) in terms of ψ_n^α and $\dot{\psi}_n^\alpha$ in the following way:

$$S := \sum_n \{i(\psi_n^{*\alpha}\dot{\psi}_n^{\alpha}) - 2\psi_n^{*\alpha} H_{\alpha\beta}\psi_n^{\beta}\} \ . \tag{2.37}$$

Let us call the "integrand" of the above equation the discrete Lagrangian (L) of the system. It is a function just of ψ_n^{α}, $\dot{\psi}_n^{\alpha}$, $\psi_n^{*\alpha}$ and $\dot{\psi}_n^{*\alpha}$. It is easy to see that eqs.(2.34) and (2.35) are just the following discrete Euler-Lagrange equations (introducing the notation $D(O_n^{\alpha}) = \dot{O}_n^{\alpha}$):

$$D(\delta_{D(\psi_n^{*\alpha})}L) - \delta_{\psi_n^{*\alpha}}L = 0 \ , \tag{2.38}$$

$$D(\delta_{D(\psi_n^{\alpha})}L) - \delta_{\psi_n^{\alpha}}L = 0 \ , \tag{2.39}$$

Furthemore, recalling the definition (2.29) of $C_{\hat{G}}(n,m)$, we observe that the "integrand" in eq.(2.37) is $C_{iD-\hat{H}}(n,n)$. Thus we have $L = C_{iD-\hat{H}}(n,n)$, i.e. the Lagrangian of the HCA.

2.7 Introducing the time scale

At this point, we want to introduce a physical time scale l in our automaton (l has the dimension of time). To do this, we will substitute every quantity depending on $n \in \mathbb{Z}$ with a quantity depending on ln, $l \in \mathbb{R}$, $n \in \mathbb{Z}$.

We consider an HCA with variables $x^{\alpha}(ln)$ and their "conjugated momenta" $p^{\alpha}(ln)$, where α is a multi index and $x^{\alpha}(ln), p^{\alpha}(ln) \in \mathbb{R}$. Then we introduce the quantities (using the summation convention for repeated greek indices):

$$A(ln) := 2clH(ln) \ , \tag{2.40}$$

$$H(ln) := \frac{1}{2}S_{\alpha\beta}\left(p^{\alpha}(ln)p^{\beta}(ln) + x^{\alpha}(ln)x^{\beta}(ln)\right) + A_{\alpha\beta}p^{\alpha}(ln)x^{\beta}(ln) \ , \tag{2.41}$$

where c is a constant, $\hat{S} = \{S_{\alpha\beta}\}$ is a symmetric matrix, $\hat{A} = \{A_{\alpha\beta}\}$ is an antisymmetric matrix and we multiplied $H(ln)$ with l to give it the right dimension $[H(ln)] = [time]^{-1}$. The factor 2 in eq.(2.40) is for convenience. We use as a definition for the action S:

$$S := \sum_n [(p^\alpha(ln) + p^\alpha(l(n-1)))\Delta x^\alpha(ln) - A(ln)] \ , \tag{2.42}$$

where $\Delta x^\alpha(ln) = x^\alpha(l(n)) - x^\alpha(l(n-1))$.

As before, we define the variation of a function $g(f(ln))$ as:

$$\delta g(f(ln)) := \frac{1}{2}[g(f(ln) + \delta f(ln)) - g(f(ln) + \delta f(ln))] \ . \tag{2.43}$$

Then we apply **Postulate A** to obtain the updating equations:

$$\dot{x}^\alpha(ln) = c\left(S_{\alpha\beta}p^\beta(ln) + A_{\alpha\beta}x^\beta(ln)\right) \ , \tag{2.44}$$

$$\dot{p}^\alpha(ln) = -c\left(S_{\alpha\beta}x^\beta(ln) - A_{\alpha\beta}p^\beta(ln)\right) \ , \tag{2.45}$$

where now $\dot{O} := (O(l(n+1)) - O(l(n-1))/2l)$. As before, we can make the change of variables from $x^\alpha(ln)$ and $p^\alpha(ln)$ to $\psi^\alpha(ln) = x^\alpha(ln) + ip^\alpha(ln)$ and its complex conjugate obtaining the updating equation for $\psi^\alpha(ln)$:

$$\dot{\psi}^\alpha(nl) = -icH_{\alpha\beta}\psi^\beta(nl) \ , \tag{2.46}$$

plus its complex conjugate for $\psi^{*\alpha}(ln)$. Moreover, **Theorem A** is still valid. So, if a matrix $\hat{G}$ commutes with $\hat{H}$, we have the conservation law:

$$\psi^{*\alpha}(ln)G_{\alpha\beta}\dot{\psi}^\beta(ln) + \dot{\psi}^{*\alpha}(ln)G_{\alpha\beta}\psi^\beta(ln) = 0 \ . \tag{2.47}$$

From which we can extract the conserved two-time symmetric function $C_{\hat{G}}(ln-l), ln)$ defined as:

$$2C_{\hat{G}}(ln-l, ln) := \psi^{*\alpha}(ln-l)G_{\alpha\beta}\psi^\beta(ln) + c.c. = const \ . \tag{2.48}$$

We can also write in this case the general solution for $\psi^\alpha(ln)$ that is:

$$\psi^\alpha(ln) = -i^n\left[(U_{n-2}(-cl\hat{H}))_{\alpha\beta}\psi^\beta(0) + i(U_{n-1}(-cl\hat{H}))_{\alpha\beta}\psi^\beta(l)\right] \ . \tag{2.49}$$

The introduction of the time scale l has been done, in order to have the possibility of considering the continuum limit $l \to 0$ and, thus, to see which features of quantum mechanics can be recovered in this limit from the HCA. This will be studied in detail in next chapter.

Chapter 3

States, Observables and the Continuum Limit

3.1 Introduction

What we want to do now is to study the space of states and the observables of our HCA, i.e., the pertinent algebraic structures. In the first section, we will not make any assumption on the space of states structure and we will consider as observables every constant, linear, and quadratic function of the variables, which gives us a structure very similar to that of classical Hamiltonian systems.

Then, we will consider in this “classical” case the limit $l \to 0$. We will see in the next section that in this limit our system becomes an ordinary Hamiltonian system with Poisson brackets and Hamilton's equations of motion. The continuum system is a kind of oscillator system with a kinetic energy which is not diagonal. So one may wonder whether it is of practical interest in physics.

However, it has been shown by Heslot in [18] that quantum mechanics can be formulated in a generalized classical Hamiltonian form, finding an Hamiltonian system with a space of states that has a complementary structure in addition to the Poisson brackets. With this in mind, we can look for some structure to introduce in the HCA which in the continuum limit will go into the quantum mechanical complementary structure. We will indeed find it and then recover, in the continuum limit, all the quantum mechanical features except for the Born rule that cannot be introduced at the HCA level.

3.2 The structure of the space of states

In this section, we will make use of the variables $\{x^\alpha(t_n), p^\alpha(t_n)\}$, with $t_n = ln$.

Let us look at the space of states of our HCA. Because the discrete updating equations are of second order the state of the system at time n is characterized by the two sets of variables $\{x^\alpha(t_n), p^\alpha(t_n)\}$ and $\{x^\alpha(t_n - l), p^\alpha(t_n - l)\}$.

As a first attempt, we want to consider the variables of our system in the same way as classical variables of a Hamiltonian system. So, using the terminology of classical systems, we will call observables all the constant linear and quadratic real valued regular functions of the state variables. We remark that we consider functions only up to quadratic order, because the definition of the variation allows us to work easily just with these functions, cf. below.

Note that we can try to define for our Cellular Automaton a structure similar to the Poisson bracket structure of classical mechanics. In fact, once we defined the variation, given two functions $f(t_n)$ and $g(t_n)$ of the Cellular Automaton variables, we can define for each time t_n the following operation between them:

$$\begin{aligned} &\{f(t_n), g(t_n)\}_{(x(t_n),p(t_n))} \equiv \{f(t_n), g(t_n)\}_{t_n} := \\ &\left[\frac{\delta_{x^\alpha(t_n)} f(t_n)}{\delta x^\alpha(t_n)} \frac{\delta_{p^\alpha(t_n)} g(t_n)}{\delta p^\alpha(t_n)} - \frac{\delta_{p^\alpha(t_n)} f(t_n)}{\delta p^\alpha(t_n)} \frac{\delta_{x^\alpha(t_n)} g(t_n)}{\delta x^\alpha(t_n)}\right] , \end{aligned} \tag{3.1}$$

where $x(t_n)$ and $p(t_n)$ are collective dynamical variables and the summation convention for repeated greek indices is used.

In defining the brackets above we have to remember that we are considering arbitrary variations, so the brackets defined in eq.(3.1) will be similar to Poisson brackets only for constant, linear, and quadratic functions. However, with this definition, we had that the time evolution does not preserve the bracket structure we just defined. In fact, we have that (calling $T_{t_m}(f(t_n)) := f(t_n + t_m)$ the m steps time evolution of the function $f(t_n)$):

$$T_{t_m}(\{f(t_n), g(t_n)\}_{(x(t_n),p(t_n))}) \neq \{T_{t_m}(f(t_n)), T_{t_m}(g(t_n))\}_{(x(t_n),p(t_n))} \,. \tag{3.2}$$

This is due to the definition of the discrete "time derivative". In particular, we have to notice that our updating equations need two initial conditions, so they are second order equations.

Despite this, the brackets defined above have some use. For example, we can use them to write the updating equations in this way:

$$\dot{x}^{\alpha}(t_n) = \{x^{\alpha}(t_n), H(t_n)\}_{t_n} \ , \tag{3.3}$$

$$\dot{p}^{\alpha}(t_n) = -\{p^{\alpha}(t_n), H(t_n)\}_{t_n} \ , \tag{3.4}$$

and also the conservation laws of **Theorem A** as follows. Consider the two-times bilinear function of the variables $(x(t_n), p(t_n))$ and $(x(t_m), p(t_m))$:

$$G(t_n, t_m) := G(x(t_n), p(t_n); x(t_m), p(t_m)) \ , \tag{3.5}$$

where bilinearity means that $G(x(t_n) + ay(t_n), p(t_n) + aq(t_n); x(t_m), p(t_m)) = G(x(t_n), p(t_n); x(t_m), p(t_m)) + aG(y(t_n), q(t_n); x(t_m), p(t_m))$ and the same for the second argument. Then, if $G(t_n, t_n) := G(t_n)$ is such that $\{G(t_n), H(t_n)\}_{t_n} = 0$, we have the conservation law:

$$G(x(t_n), p(t_n); \dot{x}(t_n), \dot{p}(t_n)) = 0 \ . \tag{3.6}$$

And also the conserved quantity:

$$G(x(t_n), p(t_n); x(t_n - l), p(t_n - l)) = const \ . \tag{3.7}$$

We can easily see which form has to take $G(x(t_n), p(t_n); x(t_m), p(t_m))$ by considering **Theorem A** and writing there ψ in terms of x and p (remember that $\psi = x + ip$). In fact, we have that:

$$G(x(t_n), p(t_n); x(t_m), p(t_m)) = \psi^{*\alpha}(t_n) G_{\alpha\beta} \psi^{\beta}(t_m) + \psi^{*\alpha}(t_m) G_{\alpha\beta} \psi^{\beta}(t_n) \ , \tag{3.8}$$

where $G_{\alpha\beta}$ is a Hermitean matrix and $\psi^{*\alpha}(t_n)$ and $\psi^{\alpha}(t_n)$ must be considered as functions of $x^{\alpha}(t_n)$ and $p^{\alpha}(t_n)$. We can rewrite $G_{\alpha\beta}$ as $G^S_{\alpha\beta} + iG^A_{\alpha\beta}$, where $G^S_{\alpha\beta}$ is a symmetric matrix and $G^A_{\alpha\beta}$ is an antisymmetric matrix. Then we have:

$$G(x(t_n), p(t_n); x(t_m), p(t_m)) =$$
$$2G^S_{\alpha\beta}\left(x^\alpha(t_n)x^\beta(t_m) + p^\alpha(t_n)p^\beta(t_m)\right) - 2G^A_{\alpha\beta}\left(x^\alpha(t_n)p^\beta(t_m) - p^\alpha(t_n)x^\beta(t_m)\right) \ . \tag{3.9}$$

Note also that trivially we have the familiar looking relations $\{x^\alpha(t_n), p^\beta(t_n)\}_{t_n} = \delta_{\alpha\beta}$ and $\{x^\alpha(t_n), x^\beta(t_n)\}_{t_n} = \{p^\alpha(t_n), p^\beta(t_n)\}_{t_n} = 0$.

In classical mechanics Poisson brackets are important because their structure is preserved by the so called canonical transformations: transformations that do not change the physics of the system. So let us see if there are transformations of the variables of our HCA that preserve the structure of the brackets we defined. Consider an infinitesimal transformation of the variables of this kind:

$$\begin{aligned} x^\alpha(t_n) \to x'^\alpha(t_n) &= x^\alpha(t_n) + \{x^\alpha(t_n), g(t_n)\}_{t_n}\delta\theta \ , \\ p^\alpha(t_n) \to p'^\alpha(t_n) &= p^\alpha(t_n) + \{p^\alpha(t_n), g(t_n)\}_{t_n}\delta\theta \ , \end{aligned} \tag{3.10}$$

where $\delta\theta$ is an infinitesimal real parameter. The transformation (3.10) is such that we have:

$$\{x'^\alpha(t_n), p'^\beta(t_n)\}_{(x^\alpha(t_n),p^\alpha(t_n))} = \delta_{\alpha\beta} \ , \tag{3.11}$$

and

$$\{x'^\alpha(t_n), x'^\beta(t_n)\}_{(x^\alpha(t_n),p^\alpha(t_n))} = \{p'^\alpha(t_n), p'^\beta(t_n)\}_{(x^\alpha(t_n),p^\alpha(t_n))} = 0 \ . \tag{3.12}$$

So we have some sort of canonical transformations which preserve the bracket structure, and which commute with the time evolution, Note, however, that the time evolution is not described by one of these transformations, since the updating equations contain a t_{n-1} instead of t_n ($x^\alpha(t_n) \to x'^\alpha(t_n) = x^\alpha(t_{n+1}) = x^\alpha(t_{n-1}) + \{x^\alpha(t_n), H(t_n)\}_{t_n}\delta\theta$). This will be illustrated by the following example.

3.2.1 A N=2 example

Let us take a look at an explicit example for a two-dimensional HCA.

We consider a two dimensional system with variables $x(t_n) = (x^1(t_n), x^2(t_n))$ and $p(t_n) = (p^1(t_n), p^2(t_n)$, and the two matrices $\hat{S}$, the symmetrical one, and $\hat{A}$ the antisymmetrical one:

$$\hat{S} = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} , \tag{3.13}$$

$$\hat{A} = \begin{pmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{pmatrix} . \tag{3.14}$$

Following the definition (2.42), the action of the system is of the form:

$$S := \sum_n \left[(p^1(t_n) + p^1(t_n - l)\Delta x^1(t_n)) + (p^2(t_n) + p^2(t_n - l)\Delta x^2(t_n)) - 2clH(t_n) \right] , \tag{3.15}$$

where $\Delta x(t_n) = x(t_n) - x(t_n - l)$ and:

$$H(t_n) = \frac{1}{2} \sum_{\alpha,\beta=1}^{2} s_{\alpha\beta} \Big(p^\alpha(ln)p^\beta(ln) + x^\alpha(ln)x^\beta(ln) \Big) + a_{\alpha\beta} p^\alpha(ln)x^\beta(ln) . \tag{3.16}$$

From which we derive the updating equations, similarly as before:

$$\begin{aligned}
\dot{x}^1(t_n) &= \tfrac{x^1(t_n+l) - x^1(t_n-l)}{2l} = c\left(\tfrac{1}{2}s_{11}p^1(t_n) + \tfrac{1}{2}s_{12}p^2(t_n) + a_{12}x^2(t_n)\right) , \\
\dot{x}^2(t_n) &= \tfrac{x^2(t_n+l) - x^2(t_n-l)}{2l} = c\left(\tfrac{1}{2}s_{12}p^1(t_n) + \tfrac{1}{2}s_{22}p^2(t_n) - a_{12}x^1(t_n)\right) , \\
\dot{p}^1(t_n) &= \tfrac{p^1(t_n+l) - p^1(t_n-l)}{2l} = -c\left(\tfrac{1}{2}s_{11}x^1(t_n) + \tfrac{1}{2}s_{12}x^2(t_n) - a_{12}p^2(t_n)\right) , \\
\dot{p}^2(t_n) &= \tfrac{p^2(t_n+l) - p^2(t_n-l)}{2l} = -c\left(\tfrac{1}{2}s_{12}x^1(t_n) + \tfrac{1}{2}s_{22}x^2(t_n) + a_{12}p^1(t_n)\right) .
\end{aligned} \tag{3.17}$$

As we already said, the state of the system at time t_n is characterized by the two sets of variables: $\{x(t_n), p(t_n)\}$ and $\{x(t_n - l), p(t_n - l)\}$.

Now consider, for example, two functions $F(t_n) = 2x^1(t_n)p^1(t_n)$, $G(t_n) = x^2(t_n)p^1(t_n)$; the brackets defined in eq.(3.1) read:

$$\{F(t_n), G(t_n)\}_{(x(t_n),p(t_n))} =$$
$$\sum_{\alpha=1}^{2}\left[\frac{\delta_{x^\alpha(t_n)}F(t_n)}{\delta x^\alpha(t_n)}\frac{\delta_{p^\alpha(t_n)}G(t_n)}{\delta p^\alpha(t_n)} - \frac{\delta_{p^\alpha(t_n)}F(t_n)}{\delta p^\alpha(t_n)}\frac{\delta_{x^\alpha(t_n)}G(t_n)}{\delta x^\alpha(t_n)}\right] = 2p^1(t_n)x^2(t_n)\ . \tag{3.18}$$

Let us see that an inequality as (3.2), for $m = 1$, applies for $F(t_n)$ and $G(t_n)$. We have to consider the time evolution of their bracket at time t_n and the bracket at time t_n of their time evolution. The first case gives:

$$T_1\left(\{F(t_n), G(t_n)\}_{(x(t_n),p(t_n))}\right) = 2p^1(t_n+l)x^2(t_n+l) =$$
$$2\left[p^1(t_n-l) - 2lc\left(\tfrac{1}{2}s_{11}x^1(t_n) + \tfrac{1}{2}s_{12}x^2(t_n) - a_{12}p^1(t_n)\right)\right]\cdot \tag{3.19}$$
$$\left[x^2(t_n-l) + 2lc\left(\tfrac{1}{2}s_{12}p^1(t_n) + \tfrac{1}{2}s_{22}p^2(t_n) - a_{12}x^1(t_n)\right)\right]\ ,$$

where the dot at the end of the second line stands for the usual product. While, for the second case, we have that $\{T_1(F(t_n)), T_1(G(t_n))\}_{(x(t_n),p(t_n))}$ is a function only of $x(t_n)$ and $p(t_n)$ (it does not contain terms with $x(t_n - l)$ and $p(t_n - l)$), so it is easy to see that it is different from (3.19). This illustrates our earlier statement that time evolution, here, is not described by a canonical transformation.

Next let us take an explicit Hamiltonian function, e.g.:

$$\hat{S} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\ , \qquad \hat{A} = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}\ , \tag{3.20}$$

and try to find a function $G(x(t_n), p(t_n); x(t_n), p(t_n))$ that has the bracket we defined before with the Hermitean matrix $\hat{S} + i\hat{A}$ equal to zero, in order to see that it gives a conservation law. As we said in the previous section, G has to be of the form of eq.(3.9). We take the two matrices $\hat{G}^S$ and $\hat{G}^A$ to be:

$$\hat{G}^S = \begin{pmatrix} 5/3 & 0 \\ 0 & 4/3 \end{pmatrix}\ , \qquad \hat{G}^A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ . \tag{3.21}$$

Now it is easy to see that indeed $\{H(t_n), G(t_n)\}_{(x(t_n),p(t_n))} = 0$ for every n, so we have:

$$\tfrac{5}{3}\left(x^1(t_n)\dot{x}^1(t_n) + p^1(t_n)\dot{p}^1(t_n)\right) + \tfrac{4}{3}\left(x^2(t_n)\dot{x}^2(t_n) + p^2(t_n)\dot{p}^2(t_n)\right) +$$
$$-2\left(x^1(t_n)\dot{p}^2(t_n) - x^2(t_n)\dot{p}^1(t_n) - p^1(t_n)\dot{x}^2(t_n) + p^2(t_n)\dot{x}^1(t_n)\right) = 0\ , \tag{3.22}$$

From the above equation, we obtain the conserved quantity:

$$\frac{5}{3}\left(x^1(t_n)x^1(t_n-l)+p^1(t_n)p^1(t_n-l)\right)+\frac{4}{3}\left(x^2(t_n)x^2(t_n-l)+p^2(t_n)p^2(t_n-l)\right)+$$
$$-2\left(x^1(t_n)p^2(t_n-l)-x^2(t_n)p^1(t_n-l)-p^1(t_n)x^2(t_n-l)+p^2(t_n)x^1(t_n-l)\right)=const\;. \tag{3.23}$$

Finally, we want to show that there are, however, transformations that do preserve the bracket structure. We consider a rotation by a finite angle of the $x(t_n)$, and $p(t_n)$ to show that it preserves the brackets we defined and also that it commutes with the time evolution.

We start with an infinitesimal rotation, which can be written in the form of eqs.(3.10), with $g(t_n)=x^1(t_n)p^2(t_n)-x^2(t_n)p^1(t_n)$. To handle the finite transformation, it is easier to write the transformation in the form:

$$\begin{pmatrix}x^1(t_n)\\x^2(t_n)\end{pmatrix}\to\begin{pmatrix}x'^1(t_n)\\x'^2(t_n)\end{pmatrix}=\begin{pmatrix}x^1(t_n)\cos\theta-x^2(t_n)\sin\theta\\x^2(t_n)\cos\theta+x^1(t_n)\sin\theta\end{pmatrix}\;, \tag{3.24}$$

where θ is the angle of rotation. First let us show that this transformation preserves the bracket structure for every t_n. To do this, it is sufficient to show that it preserves the value of the brackets for the variables of the system. So we have to evaluate $\{x'^\alpha(t_n),p'^\beta(t_n)\}_{(x(t_n),p(t_n))}$, $\{x'^\alpha(t_n),x'^\beta(t_n)\}_{(x(t_n),p(t_n))}$, and $\{p'^\alpha(t_n),p'^\beta(t_n)\}_{(x(t_n),p(t_n))}$.

Because x' does not depend on p and p' does not depend on x we have that $\{x'^\alpha(t_n),x'^\beta(t_n)\}_{(x(t_n),p(t_n))}=\{p'^\alpha(t_n),p'^\beta(t_n)\}_{(x(t_n),p(t_n))}=0$, as expected. Now we check the first bracket:

$$\{x'^\alpha(t_n),p'^\beta(t_n)\}_{(x(t_n),p(t_n))}=$$
$$\{x^\alpha(t_n)\cos\theta-\epsilon^{\alpha\gamma}x^\gamma(t_n)\sin\theta,p^\beta(t_n)\cos\theta-\epsilon^{\beta\delta}p^\delta(t_n)\sin\theta\}_{(x(t_n),p(t_n))}= \tag{3.25}$$
$$\delta^{\alpha\beta}(\sin^2\theta+\cos^2\theta)=\delta^{\alpha\beta}\;,$$

where $\epsilon^{\alpha\beta}$ is the antisymmetric tensor. So the bracket structure is preserved. The last thing we need to show is that rotations commute with time evolution. For simplicity, we will justify it just for a one-step evolution. Mathematically, if we call $R(x(t_n))=x'(t_n)$ and $T_1(x(t_n))=x(t_n+l)$, we need to show that $R(T_1(x(t_n)))=T_1(R(x(t_n)))$. It is easy to show this equivalence, because the rotation parameter θ does not depend on t_n.

3.3 The continuum limit of the classical HCA

Now that we have the space of states structure, we can study the limit $l \to 0$, $t_n \to t$.

Consider the action S:

$$S := \sum_n \left[\frac{1}{2} \left(p^\alpha(t_n) + p^\alpha(t_n - l) \right) \frac{\Delta x^\alpha(t_n)}{l} - A(t_n) \right] 2l \; , \tag{3.26}$$

with:

$$A(t_n) := cH(t_n) \; , \tag{3.27}$$

and

$$H(t_n) := \frac{1}{2} S_{\alpha\beta} (p^\alpha(t_n) p^\beta(t_n) + x^\alpha(t_n) x^\beta(t_n)) + A_{\alpha\beta} p^\alpha(t_n) x^\beta(t_n) \; , \tag{3.28}$$

where c is a constant, $\hat{S} = \{S_{\alpha\beta}\}$ is a symmetric matrix, $\hat{A} = \{A_{\alpha\beta}\}$ is an antisymmetric matrix. Note that these definitions are equivalent to those given in Chapter (2), Section (2.7). Then, once we apply **Postulate A**, the updating equations will be eqs. (2.44) and (2.45), and we have also the conservation laws of **Theorem A**. Now let us consider $l \equiv dt$ and the limit $dt \to 0$ with $ndt = t$. We will get for S:

$$\lim_{l \to 0} S := \int \left[p^\alpha(t) \frac{dx^\alpha(t)}{dt} - A(t) \right] 2dt \; , \tag{3.29}$$

with:

$$A(t) := cH(t) \; , \tag{3.30}$$

and

$$H(t) := \frac{1}{2} S_{\alpha\beta} (p^\alpha(t) p^\beta(t) + x^\alpha(t) x^\beta(t)) + A_{\alpha\beta} p^\alpha(t) x^\beta(t) \; . \tag{3.31}$$

The definition of the variation is that of eq.(2.43) substituting t_n with t. Then, we can apply **Postulate A** and get the Hamiltonian equations of motion:

$$\dot{x}^\alpha(t) = \frac{dH(t)}{dx^\alpha(t)} = c(S_{\alpha\beta}p^\beta(t) + A_{\alpha\beta}x^\beta(t)) \ , \tag{3.32}$$

$$\dot{p}^\alpha(t) = -\frac{dH(t)}{dp^\alpha(t)} = -c(S_{\alpha\beta}x^\beta(t) - A_{\alpha\beta}p^\beta(t)) \ , \tag{3.33}$$

with $\dot{O} = dO/dt$ the symmetric time derivative of O. Now we can consider two functions f and g of the dynamical variables and the operation $\{f, g\} = \sum_\alpha \left(\frac{\delta f}{\delta x^\alpha} \frac{\delta g}{\delta p^\alpha} - \frac{\delta f}{\delta p^\alpha} \frac{\delta g}{\delta x^\alpha} \right)$. We have two possibilities: First, we can consider arbitrary variations as in the discrete case; then we have that $\{,\}$ are Poisson brackets only for constant, linear, and quadratic functions and, because of Hamilton's equations of motion, in this limit the time evolution will preserve their structure; that is (calling $T_{t1}(f(t))$ the time translation of f):

$$T_{t_1}(\{f, g\}) = \{T_{t_1}(f), T_{t_1}(g)\} \ . \tag{3.34}$$

This holds, because in the continuum limit the "time derivative" we defined becomes the ordinary symmetric derivative.

Second, if we consider infinitesimal variations, the δ becomes a partial derivative and the Poisson brackets are the classical ones.

It is possible to write the conservation laws using the Poisson brackets, as we have done in the previous section. We get for each bilinear function of the variables at time t:

$$G(t) := G(x(t), p(t); x(t), p(t)) \ , \tag{3.35}$$

such that $\{G(t), H(t)\} = 0$, we have the conservation law:

$$\dot{G} = 0 \ . \tag{3.36}$$

And then, from this the conserved quantity $G(t)$.

So, in the continuum limit, we recovered all of the feautures of Hamiltonian systems for quadratic potentials, except for the fact that the time derivative is the symmetric one.

3.4 Complementary structure

So far, in this Chapter (3), our study aimed at reconstructing as much as possible the classical Hamiltonian dynamics, in particular in the continuum limit, beginning with a HCA. Presently, instead, we intend to go further by trying to incorporate also a complementary "pre-quantum" structure, such that quantum mechanics is recovered in the appropriate limit.

In this section we will use the variables $\{\psi^\alpha(t_n)\}$, with $t_n = ln$, in order to simplify the equations.

First of all, note that we can consider our variables ψ^α as coefficients of a complex vector in a Hilbert space $\mathcal{H}$. Then a state is characterized by two complex vectors (the one at time t_n and the one at time $t_n - l$). So up to now the space of states of our system consists of the direct sum of two complex vector spaces. Let us call V the space of states of our system and $\Psi(t_n) = (\psi_1(t_n), \psi_2(t_n))$, with $\psi_2(t_n) = \psi_1(t_n - l)$ a state of it, where $\psi_1(t_n) = \{\psi^\alpha(t_n)\}$.

The main reason of defining such a Ψ is to build an inner product structure (positive definite). This will be employed in writing the constants of motion of our HCA.

We would like to work under the following assumptions:

1 A state of the system is characterized by two Hilbert space vectors at successive times $\psi_1(t_n)$ and $\psi_2(t_n)$ or equivalently $\psi_+(t_n) = (1/2)(\psi_1(t_n) + \psi_2(t_n))$ and $\psi_-(t_n) = (1/2)(\psi_1(t_n) - \psi_2(t_n))$;

2 The space of states V is itself a Hilbert space, which is the direct sum of the two Hilbert spaces mentioned before; a state is, thus, written as $\Psi(t_n) = (\psi_1(t_n), \psi_2(t_n))$ or equivalently $\Psi'(t_n) = (\psi_+(t_n), \psi_-(t_n))$;

3 The superposition principle must hold.

The introduction of the new variables ψ_+ and ψ_- serves to have a state Ψ' that has one component (ψ_+) of order $O(1)$ and the other (ψ_-) of order $O(l)$. This will be useful for approximations considered below.

The observables will be Hermitean operators acting on V.

We will try to also define restricted observables with the following important characteristics:

(i) The restricted observables of the system are those Hermitean matrices which can be written for the state $\Psi(t_n)$ as $\hat{\mathbf{O}}^{\mathbf{G}} = \hat{\mathbf{G}}\Sigma_1$ or for the state $\Psi'(t_n)$ as $\hat{\mathbf{O}}^{\mathbf{G}} = \hat{\mathbf{G}}\Sigma_3$, where:

$$\Sigma_1 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} ,$$

$$\Sigma_3 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} ,$$

and

$$\hat{\mathbf{G}} = \begin{pmatrix} \hat{G} & 0 \\ 0 & \hat{G} \end{pmatrix} .$$

The observables form a C*-algebra.

The requirement of a C*-algebra here, is, of course, motivated by having precisely this structure in the corresponding quantum mechanical setting [19].

Note that the operators Σ_1 and Σ_3 act as metric operators and, with them and the inner product of the Hilbert space, we can define an indefinite inner product turning V into a Krein space; for a mathematical definition and further discussion see reference [20]. The indefinite inner product can be written as:

$$\langle \Psi, \Phi \rangle_K = \langle \Psi, \Sigma_1 \Phi \rangle , \tag{3.37}$$

or in terms of Ψ' as:

$$\langle \Psi', \Phi' \rangle_K = \langle \Psi', \Sigma_3 \Phi' \rangle . \tag{3.38}$$

3.4.1 Loss of linearity

Before going on, a remark is in order. If we take a look at the continuum limit, we can see that the conservation laws of our system become very similar to the quantum mechanical conservation laws, and the time evolution becomes unitary. In fact, we have that:

(a) as shown in [16], the solution for $\psi^\alpha(t_n)$ given $\psi^\alpha(t_k)$ and $\psi^\alpha(t_k + l)$ is:

$$\psi^\alpha(t_n) = -i^{n-k}\left((U_{n-k-2}[-cl\hat{H}])_{\alpha\beta}\psi^\beta(t_k) + i(U_{n-k-1}[-cl\hat{H}])_{\alpha\beta}\psi^\beta(t_k+l)\right) , \tag{3.39}$$

where $U_i[-cl\hat{H}]$ is the i^{th} Chebyshev polynomial of the second kind.

Then if $\psi^\alpha(t_n) - \psi^\alpha(t_n - l) \propto l$, in the limit $l \to 0$, $t_n = t$, $t_k = t_0$ we get:

$$\psi^\alpha(t) = e^{-icH_{\alpha\beta}(t-t_0)}\psi^\beta(t_0) .$$

(b) from **Theorem A**, if $[\hat{G}, \hat{H}] = 0$, then $C_{\hat{G}}(t_n, t_n-l)$ does not depend on n, and in the limit $l \to 0$, $t_n = t$, we have $C_{\hat{G}}(t_n, t_n - l) \to \psi^{*\alpha}(t)G_{\alpha\beta}\psi^\beta(t)$, which is conserved. This is precisely the quantum mechanical conservation law for $\langle\psi|\hat{G}|\psi\rangle$, written in terms of the components of $|\psi\rangle$. In particular, this holds for $\hat{G} = \hat{\mathbb{I}}$, from which we obtain the norm conservation for $|\psi\rangle$.

Note that, in general, $C_{\hat{\mathbb{I}}}(t_n, t_n - l)$ can be bigger, less then, or equal zero, even if in the limit we have $C_{\hat{\mathbb{I}}}(t, t) \geq 0$, with $C_{\hat{\mathbb{I}}}(t, t) = 0$ if and only if $\psi^\alpha = 0 \; \forall\alpha$.

With this in mind, one could try to build a structure similar to that of quantum mechanics for the HCA. In particular it seems to be straightforward to restrict the space of states such that $C_{\hat{\mathbb{I}}}(t_n, t_n - l) \geq 0$, on the restricted space. In the following, we will see that this restriction cannot be implemented without losing the linearity, and hence the superposition principle for the states.

As we have seen in the previous section, a state of the system at time t_n is characterized by the two vectors $\psi^\alpha(t_n)$ and $\psi^\alpha(t_n - l)$. To show that linearity is lost when restricting the space, it is better to change variables, and use instead of $\Psi(t_n) = (\psi_1(t_n), \psi_2(t_n))$ the state $\Psi'(t_n) = (\psi_+(t_n), \psi_-(t_n))$, where $\psi_+(t_n) = \psi_1(t_n) + \psi_2(t_n)$ and $\psi_-(t_n) = \psi_1(t_n) - \psi_2(t_n)$. Now we can consider the condition:

$$C_{\hat{\mathbb{I}}}(t_n, t_n - l) \geq 0 , \tag{3.40}$$

that can be rewritten in terms of $\psi_+(t_n)$ and $\psi_-(t_n)$ as:

$$\psi_+^{*\alpha}\psi_+^\alpha - \psi_-^{*\alpha}\psi_-^\alpha \geq 0 . \tag{3.41}$$

We can draw a two dimensional figure in which we plot ψ_+ against ψ_-, in order to show the main characteristics of the space selected by the condition above. We can see from

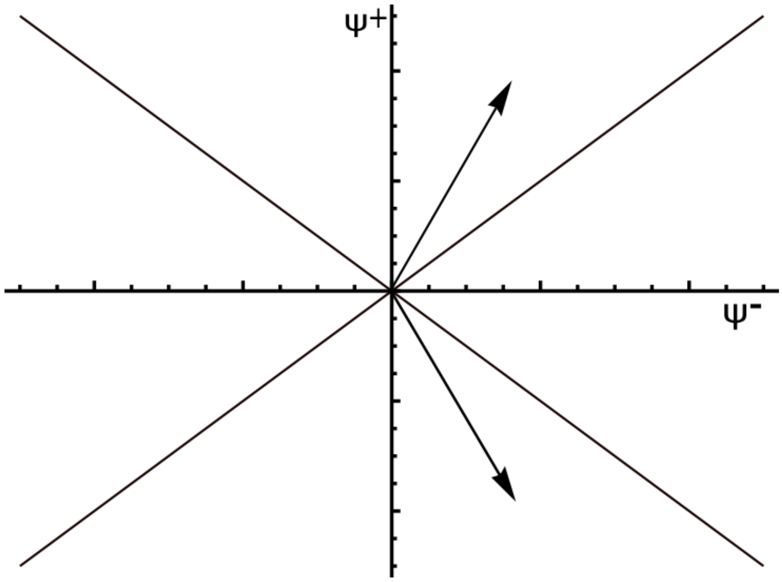


FIGURE 3.1: This figure represents the situation we would have in the space of states of the HCA, if the variables would be real and one-dimensional. On the y-axis there is ψ_+ and on the x-axis there is ψ_-. A vector with origin on $(0,0)$ represents a state. We can see that if we sum the two vectors shown in the figure, we get a vector which has null ψ_+ component, and so it is outside the region we wanted to select.

fig. 3.1 that the condition (3.41) selects the two spaces inside the "light cone" above and below the straight line $\psi_+ = 0$.

Now it is easy to see that if we choose two vectors, as in fig. 3.1, their linear sum lies outside the "light cone" and so it does not satisfy eq.(3.41). This observation invalidates the obvious attempt to restrict the state space according to the positivity condition (3.40). Therefore, we will keep the whole space of states, even though in doing this we will have the problem of giving an interpretaions to states with negative two-points correlation function.

3.4.2 V as a Hilbert space

We want to study the space of states V as a Hilbert space with the inner product defined by:

$$\langle(\psi_1(t_n), \psi_2(t_n)), (\phi_1(t_n), \phi_2(t_n))\rangle := \langle\psi_1(t_n), \phi_1(t_n)\rangle + \langle\psi_2(t_n), \phi_2(t_n)\rangle \; . \tag{3.42}$$

It is easy to see that we can write the quantities $C_{\hat{G}}(t_n, t_n - l)$ using the inner product and introducing the operator Σ_1 in V such that

$$\Sigma_1(\psi_1(t_n), \psi_2(t_n)) = (\psi_2(t_n), \psi_1(t_n)) \ , \tag{3.43}$$

so Σ_1 is of the kind of the first Pauli matrix:

$$\Sigma_1 = \begin{pmatrix} 0 & \hat{\mathbb{I}} \\ \hat{\mathbb{I}} & 0 \end{pmatrix} \ . \tag{3.44}$$

Now let us consider the following Hermitean matrix:

$$\hat{\mathbf{O}}^{\mathbf{G}} := \begin{pmatrix} 0 & \hat{G} \\ \hat{G} & 0 \end{pmatrix} \ , \tag{3.45}$$

where $\hat{G}$ is Hermitean. If $\hat{\mathbf{O}}^{\mathbf{F}}$ is a matrix of this kind and $\hat{F}$ commutes with the Hamiltonian $\hat{H}$, then the conserved quantity $C_{\hat{F}}(t_n, t_n - l)$ can be written as:

$$\begin{gathered} C_{\hat{F}}(t_n, t_n - l) = \langle (\psi_1(t_n), \psi_2(t_n)), \tfrac{\hat{\mathbf{O}}^{\mathbf{F}}}{2}(\psi_1(t_n), \psi_2(t_n)) \rangle = \\ \tfrac{1}{2}\left(\langle \psi_1(t_n), \hat{F}\psi_2(t_n) \rangle + \langle \psi_2(t_n), \hat{F}\psi_1(t_n) \rangle \right) \ . \end{gathered} \tag{3.46}$$

Now we are ready to define the observables $\hat{\mathbf{G}}$ of our system. In particular, we want to restrict our observables to Hermitean operators.

Once we have defined the observables, we have to study their algebraic properties, in order to examine the possibility to follow the quantum mechanical construction.

3.4.2.1 Algebra of the observables

Consider now the space of the observables, which are defined as Hermitean operators. They form a vector space $\mathcal{U}$ over the field $\mathbb{C}$. In fact, if we consider the usual addition of matrices we have that the sum of two observables is an observable itself, and it satisfies the axioms of vector spaces. Given three observables $\hat{\mathbf{G}}$, $\hat{\mathbf{F}}$, and $\hat{\mathbf{H}}$, and two complex numbers α, β we have that the following requirements are satisfied:

Associativity of addition: $\hat{\mathbf{G}} + (\hat{\mathbf{F}} + \hat{\mathbf{H}}) = (\hat{\mathbf{G}} + \hat{\mathbf{F}}) + \hat{\mathbf{H}}$;

Commutativity of addition: $\hat{\mathbf{G}} + \hat{\mathbf{F}} = \hat{\mathbf{F}} + \hat{\mathbf{G}}$;

Identity element of addition: there exist an element $\mathbf{0} \in \mathcal{U}$, called the zero vector, such that $\hat{\mathbf{G}} + \mathbf{0} = \hat{\mathbf{G}}$ for all $\hat{\mathbf{G}} \in \mathcal{U}$;

Inverse element of addition: for every $\hat{\mathbf{O}}^{\mathbf{G}}$ there exists an element $-\hat{\mathbf{G}}$, called the additive inverse of $\hat{\mathbf{G}}$, such that $\hat{\mathbf{G}} + (-\hat{\mathbf{G}}) = \mathbf{0}$;

Compatibility of scalar multiplication with field multiplication: $\alpha(\beta\hat{\mathbf{G}}) = (\alpha\beta)\hat{\mathbf{G}}$;

Identity element of scalar multiplication: $1\hat{\mathbf{G}} = \hat{\mathbf{G}}$, where 1 is the usual unity in $\mathbb{C}$.

Distributivity of scalar multiplication with respect to vector addition: $\alpha(\hat{\mathbf{G}} + \hat{\mathbf{F}}) = \alpha\hat{\mathbf{G}} + \alpha\hat{\mathbf{F}}$;

Distributivity of scalar multiplication with respect to field addition: $(\alpha + \beta)\hat{\mathbf{G}} = \alpha\hat{\mathbf{G}} + \beta\hat{\mathbf{G}}$.

Furthermore, we can consider the usual product between operators. It is associative and distributive; in fact, it satisfies:

(i) $\hat{\mathbf{G}}(\hat{\mathbf{F}}\hat{\mathbf{H}}) = (\hat{\mathbf{G}}\hat{\mathbf{F}})\hat{\mathbf{H}}$;

(ii) $\hat{\mathbf{G}}(\hat{\mathbf{F}} + \hat{\mathbf{H}}) = \hat{\mathbf{G}}\hat{\mathbf{F}} + \hat{\mathbf{G}}\hat{\mathbf{H}}$;

(iii) $\alpha\beta(\hat{\mathbf{G}}\hat{\mathbf{F}}) = (\alpha\hat{\mathbf{G}})(\beta\hat{\mathbf{F}})$;

but obviously it is not commutative.

With the product defined above we have the closure of the observables under the commutator. Consider two observables $\hat{\mathbf{G}}$ and $\hat{\mathbf{F}}$. Their commutator is defined as usual:

$$[\hat{\mathbf{F}}, \hat{\mathbf{G}}] = \hat{\mathbf{F}}\hat{\mathbf{G}} - \hat{\mathbf{G}}\hat{\mathbf{F}}\ , \tag{3.47}$$

and because $\hat{\mathbf{F}}$ and $\hat{\mathbf{G}}$ are Hermitean, so it is their commutator.

There is an involution mapping of the algebra of the observables, a mapping $\hat{\mathbf{G}} \in \mathfrak{U} \to \hat{\mathbf{G}}^\dagger$ with the following properties:

A $\hat{\mathbf{G}}^{\dagger\dagger} = \hat{\mathbf{G}}$;

B $(\hat{\mathbf{G}}\hat{\mathbf{F}})^\dagger = \hat{\mathbf{F}}^\dagger\hat{\mathbf{G}}^\dagger$;

C $(\alpha\hat{\mathbf{G}} + \beta\hat{\mathbf{F}})^\dagger = \alpha^*\hat{\mathbf{G}}^\dagger + \beta^*\hat{\mathbf{F}}^\dagger$.

Therefore, the algebra of observables is a *-algebra.

Next, we have to find a candidate for the norm. In quantum mechanics the norm is defined with the help of the states that are normalized positive functionals on the algebra of the quantum mechanical observables. We have to notice that, the states of the HCA are not normalized positive functionals on the algebra of the observables, so we have to define the norm for the observables in a slightly different manner w.r.t. the quantum case:

$$\|\hat{\mathbf{G}}\| = \sup_{\{(\psi_1,\psi_2)\ :\ |\langle(\psi_1,\psi_2),(\psi_1,\psi_2)\rangle|=1\}} |\langle(\psi_1,\psi_2),\hat{\mathbf{G}}(\psi_1,\psi_2)\rangle| \ . \tag{3.48}$$

The above definition satisfies the following requirements:

(i) $\|\hat{\mathbf{G}}\| \geq 0$ and $\|\hat{\mathbf{G}}\| = 0$ if and only if $\hat{\mathbf{G}} = 0$;

(ii) $\|\alpha\hat{\mathbf{G}}\| = |\alpha|\|\hat{\mathbf{G}}\|$;

(iii) $\|\hat{\mathbf{G}} + \hat{\mathbf{F}}\| \leq \|\hat{\mathbf{G}}\| + \|\hat{\mathbf{F}}\|$;

(iv) $\|\hat{\mathbf{G}}\hat{\mathbf{F}}\| \leq \|\hat{\mathbf{G}}\|\|\hat{\mathbf{F}}\|$.

Since the norm is defined with the help of the usual Hilbert space inner product, $\mathcal{U}$ is complete with respect to it; moreover, the condition $\|\hat{\mathbf{G}}^\dagger\hat{\mathbf{G}}\| = \|\hat{\mathbf{G}}\|^2$ should be satisfied, so that we should indeed have obtained a C*-algebra. For further remarks on the essential role of the C*-algebras for quantum systems, see [19].

Let us check, in particular, whether the condition $\|\hat{\mathbf{G}}^\dagger\hat{\mathbf{G}}\| = \|\hat{\mathbf{G}}\|^2$ holds true. First of all, we prove that $\|\hat{\mathbf{O}}^{\mathbf{G}}\|^2 = \|(\hat{\mathbf{O}}^{\mathbf{G}})^2\|$. To do this, notice that we have :

$$\langle(\psi_1,\psi_2),(\|\hat{\mathbf{G}}\|\hat{\mathbf{I}} \pm \hat{\mathbf{G}})(\psi_1,\psi_2)\rangle \geq 0 \ , \tag{3.49}$$

so that $\|\hat{\mathbf{G}}\|\hat{\mathbf{I}} \pm \hat{\mathbf{G}}$ are positive; then so is $(\|\hat{\mathbf{G}}\|\hat{\mathbf{I}} - \hat{\mathbf{G}})(\|\hat{\mathbf{G}}\|\hat{\mathbf{I}} + \hat{\mathbf{G}})$. From this fact we get:

$$\|\hat{\mathbf{G}}\|^2 - \|(\hat{\mathbf{G}})^2\| = \langle(\psi_1,\psi_2),(\|\hat{\mathbf{G}}\|\hat{\mathbf{I}} - \hat{\mathbf{G}})(\hat{\mathbf{G}}\|\hat{\mathbf{I}} + \hat{\mathbf{G}})(\psi_1,\psi_2)\rangle \geq 0 \ , \tag{3.50}$$

which implies $\|\hat{\mathbf{G}}\|^2 \geq \|(\hat{\mathbf{G}})^2\|$. - Furthermore, $(\|\hat{\mathbf{G}}\|\hat{\mathbf{I}} - \hat{\mathbf{G}})^2$ is positive too, thus we get:

$$\|\hat{\mathbf{G}}\|^2\hat{\mathbf{I}} - 2\|\hat{\mathbf{G}}\|\hat{\mathbf{G}} \ , \tag{3.51}$$

and:

$$
\begin{gathered}
2\|\hat{\mathbf{G}}\||\langle(\psi_1,\psi_2),\hat{\mathbf{G}}(\psi_1,\psi_2)\rangle| \leq \\
\|\hat{\mathbf{G}}\|^2 + \langle(\psi_1,\psi_2),(\hat{\mathbf{G}})^2(\psi_1,\psi_2)\rangle \leq \|\hat{\mathbf{G}}\|^2 + \|(\hat{\mathbf{G}})^2\| \ ,
\end{gathered}
\tag{3.52}
$$

from which $\|\hat{\mathbf{G}}\|^2 \leq \|(\hat{\mathbf{G}})^2\|$. This, together with the inverse inequality obtained above, implies $\|\hat{\mathbf{G}}\|^2 = \|(\hat{\mathbf{G}})^2\|$. Thanks to this result and the Hermiticity of the observables, we obtain:

$$
\|\hat{\mathbf{G}}\|^2 = \|(\hat{\mathbf{G}})^2\| = \|\hat{\mathbf{G}}^\dagger\hat{\mathbf{G}}\|. \qquad \square \tag{3.53}
$$

Since we have been able to define for our *-algebra a norm, which satisfies the condition $\|\hat{\mathbf{G}}^\dagger\hat{\mathbf{G}}\| = \|\hat{\mathbf{G}}\|^2$, we have built a C*-algebra.

In the study of HCAs we will be interested mainly in observables of the kind:

$$
\hat{\mathbf{O}}^{\mathbf{G}} = \begin{pmatrix} 0 & \hat{G} \\ \hat{G} & 0 \end{pmatrix} \ , \tag{3.54}
$$

where $\hat{G}$ is an Hermitean operator. And from now on we will refer to these as restricted observables.

3.4.2.2 Example of restricted observables and their commutator

Consider an HCA the states of which are represented by:

$$
\Psi(t_n) = (\psi_1(t_n),\psi_2(t_n)) = (\psi_1^1(t_n),\psi_1^2(t_n),\psi_2^1(t_n),\psi_2^2(t_n)) \ ,
$$

where ψ_j^i are complex numbers. We consider them as coefficients of two Hilbert space vectors and we use as a definition of inner product in the Hilbert space V the one given in (3.42):

$$
\begin{gathered}
\langle\Psi(t_n),\Phi(t_n)\rangle = \langle\psi_1(t_n),\phi_1(t_n)\rangle + \langle\psi_2(t_n),\phi_2(t_n)\rangle = \\
\psi_1^{*1}(t_n)\phi_1^1(t_n) + \psi_1^{*2}(t_n)\phi_1^2(t_n) + \psi_2^{*1}(t_n)\phi_2^1(t_n) + \psi_2^{*2}(t_n)\phi_2^2(t_n) \ .
\end{gathered}
\tag{3.55}
$$

Now, as an example, consider the quantity defined in Section 3.2, eq.(3.8), with the matrix $\hat{G}$ being:

$$\hat{G} = \begin{pmatrix} 5/3 & i \\ -i & 4/3 \end{pmatrix} . \tag{3.56}$$

Then, the restricted observable related to $\hat{G}$ is $\hat{\mathbf{O}}^{\mathbf{G}}$, cf. eq.(3.54):

$$\begin{aligned} \hat{\mathbf{O}}^{\mathbf{G}} = \tfrac{1}{2} \begin{pmatrix} 5/3 & i & 0 & 0 \\ -i & 4/3 & 0 & 0 \\ 0 & 0 & 5/3 & i \\ 0 & 0 & -i & 4/3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \\ \tfrac{1}{2} \begin{pmatrix} 0 & 0 & 5/3 & i \\ 0 & 0 & -i & 4/3 \\ 5/3 & i & 0 & 0 \\ -i & 4/3 & 0 & 0 \end{pmatrix} . \end{aligned} \tag{3.57}$$

We can easily see that $\hat{\mathbf{O}}^{\mathbf{G}}$ is Hermitean.

According to eq.(3.46) we can write $C_{\hat{G}}(t_n, t_n - l)$ as:

$$\begin{aligned} C_{\hat{G}}(t_n, t_n - l) = \langle \Psi(t_n), \hat{\mathbf{O}}^{\mathbf{G}} \Psi(t_n) \rangle = \\ \tfrac{5}{3} \Re(\psi_1^{*1}(t_n)\psi_2^1(t_n)) - \Im(\psi_1^{*1}(t_n)\psi_2^2(t_n)) + \Im(\psi_1^{*2}(t_n)\psi_2^1(t_n)) + \\ \tfrac{4}{3} \Re(\psi_1^{*2}(t_n)\psi_2^2(t_n)) . \end{aligned} \tag{3.58}$$

Now, let us take a look at the commutator between two restricted observables. We will illustrate that it is not itself a restricted observable by an explicit example. Let us take two observables, $\hat{\mathbf{O}}^{\mathbf{G}}$ and $\hat{\mathbf{O}}^{\mathbf{F}}$, and evaluate the commutator:

$$[\hat{\mathbf{O}}^{\mathbf{G}}, \hat{\mathbf{O}}^{\mathbf{F}}] = (\hat{\mathbf{O}}^{\mathbf{G}} \hat{\mathbf{O}}^{\mathbf{F}}) - (\hat{\mathbf{O}}^{\mathbf{F}} \hat{\mathbf{O}}^{\mathbf{G}}) = \begin{pmatrix} [\hat{G}, \hat{F}] & 0 \\ 0 & [\hat{G}, \hat{F}] \end{pmatrix} = \hat{\mathbf{O}}^{[\mathbf{G},\mathbf{F}]} \Sigma_1 , \tag{3.59}$$

where the last equality comes from the restricted definition of observables and from the definition of Σ_1. So the commutator between two restricted observables is block diagonal and, thus, not a restricted observable. We conclude from this, that the algebra of the restricted observables is not close w.r.t. the commutator, and this is another difference between our HCAs and quantum mechanics, where the observables are closed under the commutator.

3.5 Rewriting the action in terms of the states

As we have seen in Chapter 2, we wrote the action in terms of the variables ψ^α. We also considered them as coefficients of a Hilbert space vector, however the space of states is composed of pairs of these vectors taken at consecutive times.

Having introduced in the previous sections an inner product we can rewrite the action using this and the states. A state will be denoted with the notation $\Psi(t_n) = (\psi_1(t_n), \psi_2(t_n))$ where $\psi_1(t_n) = \{\psi^\alpha(t_n)\}$ and $\psi_2(t_n) = \psi_1(t_n - l)$. We start with considering the inner product and so the state itself will be a Hilbert space vector.

We recall the notation for the inner product:

$$\langle \Psi(t_n), \Phi(t_n) \rangle = \langle \psi_1(t_n), \phi_1(t_n) \rangle + \langle \psi_2(t_n), \phi_2(t_n) \rangle \ . \tag{3.60}$$

The updating equation for one state is easy to write, in fact, it is sufficient to use the updating equation for the variables $\psi_1(t_n)$ and to introduce an auxiliary equation for the other variables $\psi_2(t_n)$. Moreover, it can be written in a compact form:

$$\Psi(t_n + l) = \left[\Sigma_1 - il \left(\hat{\mathbf{I}} + \Sigma_3 \right) \hat{\mathbf{H}} \right] \Psi(t_n) \ , \tag{3.61}$$

where:

$$\Sigma_3 = \begin{pmatrix} \hat{\mathbb{I}} & 0 \\ 0 & -\hat{\mathbb{I}} \end{pmatrix} \ . \tag{3.62}$$

We can call the operator $[\Sigma_1 - il(\hat{\mathbf{I}} + \Sigma_3)\hat{\mathbf{H}}] = \hat{\mathbf{T}}_1$ the one-step time translation operator.

Equation (3.61) corresponds to the two equations:

$$\begin{aligned} \psi_1(t_n + l) &= \psi_2(t_n) - 2il\hat{H}\psi_1(t_n) \ , \\ \psi_2(t_n + l) &= \psi_1(t_n) \ , \end{aligned} \tag{3.63}$$

Before going on and writing the action it is useful to perform a change of variables. We can sum and subtract the two eqs. in (3.63) and rewrite them in terms of the variables $\psi_+(t_n) = (1/2)(\psi_1(t_n) + \psi_2(t_n))$ and $\psi_-(t_n) = (1/2)(\psi_1(t_n) - \psi_2(t_n))$ obtaining the two equations.:

$$\begin{aligned} \psi_+(t_n + l) &= \psi_+(t_n) - il\hat{H}(\psi_+(t_n) + \psi_-(t_n)) \ , \\ \psi_-(t_n + l) &= -\psi_-(t_n) - il\hat{H}(\psi_-(t_n) + \psi_+(t_n)) \ , \end{aligned} \tag{3.64}$$

Then, we can write a state of our HCA in terms of $\psi_+(t_n)$ and $\psi_-(t_n)$ as their direct sum:

$$\Psi'(t_n) := \psi_+(t_n) \oplus \psi_-(t_n) = (\psi_+(t_n), \psi_-(t_n)) \tag{3.65}$$

To write down an action, the variation of which leads to the updating equations, we need the inner product defined in eq.(3.42). Using this and $\Psi(t_n)$, we write:

$$S := \sum_n \langle \Psi(t_n), \Sigma_1 \Psi(t_n + l) \rangle - \langle \Psi(t_n), \Sigma_1 \left[\left(\Sigma_1 - il(\hat{\mathbf{I}} + \Sigma_3)\hat{\mathbf{H}} \right) \right] \Psi(t_n) \rangle \ , \tag{3.66}$$

or in terms of $\Psi'(t_n)$:

$$S := \sum_n \langle \Psi'(t_n), \Sigma_3 \Psi'(t_n + l) \rangle - \langle \Psi'(t_n), \Sigma_3 \left[\left(\Sigma_3 - il(\hat{\mathbf{I}} + \Sigma_1)\hat{\mathbf{H}} \right) \right] \Psi'(t_n) \rangle \ . \tag{3.67}$$

Note that for $\Psi'(t_n)$ the observables become:

$$\hat{\mathbf{O}}^{\mathbf{G}} = \hat{\mathbf{G}} \Sigma_3 \ . \tag{3.68}$$

Note that if we variate the two actions (3.66) and (3.67) w.r.t. the Hermitean conjugates of the states we get the forward updating equations for the states, while if we variate them w.r.t. the states we get the backward updating equation for the Hermitean conjugates of the states.

3.5.1 Conservation laws

Before going on to discuss the continuum limit, we want to rewrite the conservation laws in the formalism we introduced in this chapter. We have already seen the form of the conserved quantities, these are observables $\hat{\mathbf{O}}^{\mathbf{G}}$ such that $\hat{G}$ commutes with $\hat{H}$. Here we want to reformulate **Theorem A**.

We can can state that:

Theorem A' If $[\hat{\mathbf{G}}, \hat{\mathbf{H}}] = 0$ (which happens if and only if $\hat{G}$ commutes with $\hat{H}$) then $\langle \Psi(t_n), \hat{\mathbf{O}}^{\mathbf{G}} \Psi(t_n) \rangle$ does not depend on t_n.

Proof. If the hypothesis is satisfied we have that:

$$\left[\mathbf{O}^{\mathbf{G}}, \hat{\mathbf{T}}_1 \right] = \begin{pmatrix} 0 & 2il\hat{H}\hat{G} \\ -2il\hat{H}\hat{G} & 0 \end{pmatrix} . \tag{3.69}$$

From which we get:

$$\hat{\mathbf{T}}_1^\dagger \hat{\mathbf{O}}^{\mathbf{G}} \hat{\mathbf{T}}_1 = \hat{\mathbf{O}}^{\mathbf{G}} , \tag{3.70}$$

that implies:

$$\langle \Psi(t_n + l), \hat{\mathbf{O}}^{\mathbf{G}} \Psi(t_n + l) \rangle = \langle \Psi(t_n), \hat{\mathbf{O}}^{\mathbf{G}} \Psi(t_n) \rangle \qquad \square \tag{3.71}$$

This formulation of **Theorem A** will be useful when we will consider the conservation laws for composite systems.

3.6 Continuum limit

Next, we want to see what happens in the $l \to 0$, $t_n = t$ limit.

Recall that the states of our HCA are $\Psi(t_n) = (\psi_1(t_n), \psi_2(t_n))$ (or equivalently $\Psi'(t_n) = (\psi_+(t_n), \psi_-(t_n - l))$, with $\psi_-(t) \propto l$). In the limit we would get $\Psi(t) = (\psi_1(t), \psi_2(t))$ (or equivalently $\Psi'(t) = (\psi(t), 0)$). Thus, considering $\Psi(t)$, we get redundant information. Therefore, after we will have seen the limit for the updating equations, the solutions, and the value of an observable on a state, we will try to cancel the redundant information.

While, if we consider $\Psi'(t)$, we get the whole information from just the first part of the state, the second being always 0. From now on, we will consider only the $\Psi(t_n)$ state.

The updating equations are eqs.(3.63) and their limit is:

$$\begin{aligned} D\psi_1(t) &= -i\hat{H}\psi_1(t) \ , \\ \psi_2(t) &= \psi_1(t) \ , \end{aligned} \tag{3.72}$$

where $DO(t) = \lim_{l\to 0}(O(t+l) - O(t-l))/2l$ is the symmetric time derivative. We see that the second of eqs.(3.72) is an identity that gives us no information, while the first one is the Schrödinger equation. Moreover, if we consider the state $\Psi'(t_n)$ we will get the usual Schrödinger equation (with the usual derivative); in fact, from the eqs. (3.64) we get to leading order in l:

$$\begin{aligned} \tfrac{d\psi_1(t)}{dt} &= -i\hat{H}\psi_1(t) \ , \\ \tfrac{d\psi_1(t)}{dt} &= -i\hat{H}\psi_1(t) \ , \end{aligned} \tag{3.73}$$

where, as usual:

$$\frac{d\psi(t)}{dt} = \lim_{l\to 0}\frac{\psi(t+l) - \psi(t)}{l} \ . \tag{3.74}$$

Note that in the limit the two updating equations for ψ_+ and ψ_- go into the one Schrödinger equation for ψ_1

Now, let us take a look at the value of our observables $\hat{\mathbf{O}}^{\mathbf{G}}$ on a state. Because of the form of the state, in the limit, we have:

$$\langle \Psi(t), \hat{\mathbf{O}}^{\mathbf{G}}\Psi(t)\rangle = \langle \psi_1(t), \hat{G}\psi_1(t)\rangle \ , \tag{3.75}$$

which equals the mean value of quantum observables on quantum states.

In order to eliminate the redundancy of information in the states, it suffices to consider only one of the two Hilbert space states, $\psi_1(t)$, and as observables we consider simply the Hermitean matrices $\hat{G}$ with their usual algebra (that with the usual product between matrices), thus recovering the structure and the time evolution of quantum mechanics.

For the time evolution, we can consider the equivalence between our updating equation in the limit $l \to 0$, $t_n = t$ and the Schrödinger equation, or the explicit limit of the solution for the updating equation of the HCA as follows:

$$\begin{gathered}\psi_1(t_n) = -i^n \left[(U_{n-2}(-cl\hat{H}))_{\alpha\beta}\psi_1^\beta(0) + i(U_{n-1}(-cl\hat{H}))_{\alpha\beta}\psi_1^\beta(l) \right] \to \\ \to_{l\to 0} e^{-i\hat{H}t}\psi_1(0)\ ,\end{gathered} \tag{3.76}$$

where we used the properties of Chebyshev polynomials.

Note that because, in the limit, the solution is given by the usual unitary transformation, the states can be normalized, consistently with the Born rule of quantum mechanics.

For more details concerning the continuum limit see [16].

Chapter 4

Composite systems of Cellular Automata

4.1 Introduction

Now we want to combine two Hamiltonian Cellular Automata with finite sets of degrees of freedom, so that we can talk about composite systems. We will study their properties and their possible interactions.

As a first step, we consider the combination of two Hamiltonian Cellular Automata without interactions. In particular, motivated by what is known in classical and quantum physics, respectively, we propose two different ways of combining two HCA:

(1) by analogy with classical mechanics, we take the Cartesian product of the sets of variables of the two systems,

(2) by analogy with quantum mechanics, we consider the tensorial product of the two Hilbert spaces associated with the two HCA.

In Section 4.2, we will study case (1) and the analogies between the HCA and classical mechanics, while, in Section 4.3, we will study case (2) and the analogies and differences with respect to quantum mechanics.

Before going on, let us spend a few words on the reasons of considering case (2). First of all, if we want to link as many aspects as possible of our Cellular Automaton with quantum mechanics, we cannot leave aside the behaviour of a composite HCA with respect to its components. Furthermore, the tensorial structure will be particularly important when introducing an analogue of the notion of a quantum measurement for the HCA.

In all of this chapter the summation convention for repeated greek indices is used, unless it is specified otherwise.

4.2 Cartesian product structure

We want to combine two Hamiltonian Cellular Automata with finite sets of degrees of freedom. We represent their states, respectively, with the "coordinates" x_n^α and $\bar{x}_n^\beta$, and with the "conjugated momenta" p_n^α, and $\bar{p}_n^\beta$, with $n \in \mathbb{Z}$, $\alpha = 1, ..., n_1$, $\beta = 1, ..., n_2$, $n_1, n_2 \in \mathbb{N}$.

We define finite differences for all the dynamical variables:

$$\Delta f_n = f_n - f_{n-1} \ , \tag{4.1}$$

where f stands for one of the dynamical variables of one of the two Hamiltonian Cellular Automata. In addition, we introduce:

$$A_n^i := 2c^i H_n^i \ , \tag{4.2}$$

$$H_n^1 := \frac{1}{2} S_{\alpha\beta}^1 (p_n^\alpha p_n^\beta + x_n^\alpha x_n^\beta) + A_{\alpha\beta}^1 p_n^\alpha x_n^\beta \ , \tag{4.3}$$

$$H_n^2 := \frac{1}{2} S_{\alpha\beta}^2 (\bar{p}_n^\alpha \bar{p}_n^\beta + \bar{x}_n^\alpha \bar{x}_n^\beta) + A_{\alpha\beta}^1 \bar{p}_n^\alpha \bar{x}_n^\beta \ , \tag{4.4}$$

where the c^i's are constants, $\hat{S}^i = \{S_{\alpha\beta}^i\}$ are symmetric matrices, and $\hat{A}^i = \{A_{\alpha\beta}^i\}$ are antisymmetric matrices, $i = 1, 2$.

We now define the two actions, similarly as before (eq.(2.33)):

$$S^1 = \sum_n [(p_n^\alpha + p_{n-1}^\alpha) \Delta x_n^\alpha - A_n^1] \ , \tag{4.5}$$

$$S^2 = \sum_n [(\bar{p}_n^\gamma + \bar{p}_{n-1}^\gamma) \Delta \bar{x}_n^\gamma - A_n^2] \ , \tag{4.6}$$

with $\alpha = 1, ..., n_1$, $\gamma = 1, ..., n_2$. The updating equations of the two Cellular Automata follow from the variation of the two actions ($\delta S^i = 0$, $i = 1, 2$) and introducing the notation $\dot{O}_n = (O_{n+1} - O_{n-1})/2$, as before, they are:

$$\begin{aligned} \dot{x}_n^\alpha &= c^1(S^1_{\alpha\beta} p_n^\beta + A^1_{\alpha\beta} x_n^\beta) \ , \\ \dot{p}_n^\alpha &= -c^1(S^1_{\alpha\beta} x_n^\beta - A^1_{\alpha\beta} p_n^\beta) \ , \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \dot{\bar{x}}_n^\gamma &= c^2(S^2_{\gamma\delta} \bar{p}_n^\delta + A^2_{\gamma\delta} \bar{x}_n^\delta) \ , \\ \dot{\bar{p}}_n^\gamma &= -c^2(S^2_{\gamma\delta} \bar{x}_n^\delta - A^2_{\gamma\delta} \bar{p}_n^\delta) \ , \end{aligned} , \tag{4.8}$$

with $\alpha, \beta = 1, ...n_1$ and $\gamma, \delta = 1, ...n_2$.

Now we want to combine the two systems in a single system with dynamics that derives from one action principle. To do this, we consider $c^1 = c^2 := c$, then we combine the two vectors x_n^α and $\bar{x}_n^\gamma$ in a single vector X_n^β, and the other two p_n^α and $\bar{p}_n^\gamma$ in P_n^β, $\alpha = 1, ..., n_1$, $\gamma = 1, ..., n_2$, $\beta = 1, ..., n_1 + n_2$. So we have:

$$\vec{X}_n = \begin{pmatrix} x_n^1 \\ ... \\ x_n^{n_1} \\ \bar{x}_n^1 \\ ... \\ \bar{x}_n^{n_2} \end{pmatrix} \ , \tag{4.9}$$

and :

$$\vec{P}_n = \begin{pmatrix} p_n^1 \\ ... \\ p_n^{n_1} \\ \bar{p}_n^1 \\ ... \\ \bar{p}_n^{n_2} \end{pmatrix} \ . \tag{4.10}$$

Let us introduce two block matrices: the symmetric one $\hat{S}$ and the antisymmetric one $\hat{A}$, as follows:

$$\hat{S} = \begin{pmatrix} \hat{S}^1 & 0 \\ 0 & \hat{S}^2 \end{pmatrix} , \tag{4.11}$$

$$\hat{A} = \begin{pmatrix} \hat{A}^1 & 0 \\ 0 & \hat{A}^2 \end{pmatrix} . \tag{4.12}$$

Furthermore, we define for the new Cellular Automaton the quantities:

$$\begin{aligned} A_n &:= cH_n , \\ H_n &:= \tfrac{1}{2} S_{\alpha\beta}(P_n^\alpha P_n^\beta + X_n^\alpha X_n^\beta) + A_{\alpha\beta} P_n^\alpha X_n^\beta . \end{aligned} \tag{4.13}$$

And we introduce an action of the kind of eq.(4.5):

$$S := \sum_n [(P_n^\alpha + P_{n-1}^\alpha)\Delta X_n^\alpha - A_n] . \tag{4.14}$$

Note that this action is additive w.r.t. the two combined HCAs. Indeed deriving the equations of motion, as before we obtain linear equations in wich the two HCAs remain decoupled in view of eqs.(4.11) and (4.12):

$$\begin{aligned} \dot{X}_n^\alpha &= c(S_{\alpha\beta} P_n^\beta + A_{\alpha\beta} X_n^\beta) , \\ \dot{P}_n^\alpha &= -c(S_{\alpha\beta} X_n^\beta - A_{\alpha\beta} P_n^\beta) , \end{aligned} \tag{4.15}$$

4.2.1 Conservation laws

We can combine the two equations of (4.15) to write:

$$\dot{\psi}_n^\alpha = -icH_{\alpha\beta}\psi_n^\beta , \tag{4.16}$$

and its complex conjugated, where $\psi_n^\alpha = X_n^\alpha + iP_n^\alpha$ and $\hat{H} = \hat{S} + i\hat{A}$. With the updating equation written as in (4.16), it is easy to see that the composite system has discrete conservation laws. In fact, for the whole system holds **Theorem A** of ch.2.

We rewrite it here for convenience.

Theorem A For any matrix $\hat{G}$ that commutes with $\hat{H}$, $[\hat{G},\hat{H}]=0$, there is a discrete conservation law:

$$\psi_n^{*\alpha} G_{\alpha\beta} \dot{\psi}_n^{\beta} + \dot{\psi}_n^{*\alpha} G_{\alpha\beta} \psi_n^{\beta} = 0 \; . \tag{4.17}$$

As a special case of **Theorem A**, we can see that eq.(4.17) holds for every matrix $\hat{G}_{block}$ of the form:

$$\hat{G}_{block} = \begin{pmatrix} \hat{G}^1 & 0 \\ 0 & \hat{G}^2 \end{pmatrix} \; , \tag{4.18}$$

with $\hat{G}^1$ wich commutes with $\hat{H}^1$ and $\hat{G}^2$ wich commutes with $\hat{H}^2$.

Moreover, because of the absence of an interaction term, we have also discrete conservation laws for the two subsystems, as expected. For subsystem 1, we have that for any $n_1 \times n_1$ matrix $\hat{G}^1$ that commutes with $\hat{H}^1$ we have:

$$\sum_{\alpha\beta=1}^{n_1} \psi_n^{*\alpha} G^1_{\alpha\beta} \dot{\psi}_n^{\beta} + \dot{\psi}_n^{*\alpha} G^1_{\alpha\beta} \psi_n^{\beta} = 0 \; , \tag{4.19}$$

and analogous conservation laws hold for the system 2.

We can introduce an interaction term modifying $\hat{H}$ in the following way:

$$\hat{H} = \begin{pmatrix} \hat{S}^1 & \hat{I} \\ \hat{I}^{+} & \hat{S}^2 \end{pmatrix} \; , \tag{4.20}$$

where $\hat{I}$ is the interaction term and $\hat{I^{+}}$ is the Hermitean conjugate of $\hat{I}$. If we do this, we have that **Theorem A** for the whole system is still valid; in particular, eq.(4.17) holds for all the matrices $\hat{G}_{block}$ mentioned before, but eq.(4.19) for subsystem 1 and the analogous one for system 2 are no more valid, in general.

4.3 A different way to combine systems

4.3.1 Tensorial product structure

In the following we want to study how we can build a tensorial product structure of the Hilbert spaces of our Hamiltonian Cellular Automata similar to that of a composite quantum mechanical system. That it is possible, in principle is a direct consequence of the algebra of observables we built in section 3.4, see what follows. Such a structure will be important to study what entanglement could mean for Cellular Automata and to implement the concept of a measurement, in a way similar to that of quantum mechanics.

We will build this structure following what has been done in [21].

As we have seen in Ch.3, our space of states can be seen as a Hilbert space, that is a vector space in which an inner product is defined.

Now consider a finite collection $\mathcal{H}_1, \mathcal{H}_2, ..., \mathcal{H}_n$ of vector spaces. Then there exists a unique vector space $\bigodot_{i=1}^{n} \mathcal{H}_i$ with the following three properties:

(i) for each family $\{h_i\}$, where $h_i \in \mathcal{H}_i$, there exist an element $\otimes_i h_i \in \bigodot_i \mathcal{H}_i$ depending multilinearly on the h_i, and all the elements in $\bigodot_i \mathcal{H}_i$ are finite linear combinations of such elements;

(ii) (Universal property) for each multilinear mapping π of the product $\bigodot_i \mathcal{H}_i$ of the $\mathcal{H}_i$ into a vector space $\mathcal{H}'$ there exist a unique linear map $\varphi : \bigodot_i \mathcal{H}_i \to \mathcal{H}'$ such that:

$$\varphi\left(\otimes_i h_i\right) = \pi\left(\{h_i\}\right) \ ,$$

for all $h_i \in \mathcal{H}_i$. Where π acts on elements of $\mathcal{H}'$.

(iii) (Associativity) for each partition $\bigcup_k I_k$ of $\{1, ..., n\}$ there exists a unique isomorphism from $\bigodot_i \mathcal{H}_i$ onto $\bigodot_k \left(\bigodot_{i \in I_k} \mathcal{H}_i\right)$ transforming $\otimes_i h_i$ into $\otimes_k \left(\otimes_{i \in I_k} h_i\right)$.

The Universal Property (ii) is the one wich makes the space $\bigodot_i \mathcal{H}_i$ unique.

Now, because our vector spaces $\mathcal{H}_i$ are Hilbert spaces we may define an inner product on $\bigodot_i \mathcal{H}_i$ by extending the following definition by linearity:

$$\left\langle \otimes_i h_i, \otimes_i {h'}_i \right\rangle = \prod_i \left\langle h_i, {h'}_i \right\rangle \ , \tag{4.21}$$

where $h_i, h'_i \in \mathcal{H}_i$. The scalar product induces a norm and the completion of $\bigodot_i \mathcal{H}_i$ in this norm is called the tensor product of the Hilbert spaces $\mathcal{H}_i$ and is denoted with $\bigotimes_i \mathcal{H}_i$.

A similar construction can be done for the vector space of observables as well.

Indeed, if $\mathcal{U}_i$ are C*-algebras, we can make $\bigodot_{i=1}^{n} \mathcal{U}_i$ a *-algebra. In general there exist more than one norm on $\bigodot_i \mathcal{U}_i$ with the C* property and the property $\|\otimes_i A_i\| = \prod_i \|A_i\|$. For applications, the most useful norm is the so called C*-norm. This is defined by taking faithful representations $(\mathcal{H}_i, \pi_i)$ of $\mathcal{U}_i$ and defining:

$$\left\| \sum_k \otimes_i A_i^{(k)} \right\| = \left\| \sum_k \otimes_i \pi_i \left(A_i^{(k)} \right) \right\| \; . \tag{4.22}$$

This norm is independent of the particular faithful representation π_i used. Again the completion of $\bigodot_i \mathcal{U}_i$ in this norm is called the C*-tensor product of the $\mathcal{U}_i$ and is denoted by $\bigotimes_{i=1}^{n} \mathcal{U}_i$. For systems with finite degrees of freedom the norm on $\bigodot_i \mathcal{U}_i$ with the C* property and the property $\| \otimes_i A_i\| = \prod_i \|A_i\|$ is unique.

Thus, we have shown in this section that we have a unique tensor product structure both for the observables (and the operators) and for the states. The importance of having found a unique tensor product structure, both, for the observables and for the states, is that now we can proceed to couple two or more systems, being assumed that it is possible (even if not necessarily straightforward, as we shall see).

4.3.2 Tensorial product structure for HCAs

Let us consider two Hamiltonian Cellular Automata with complex variables $\xi^{\alpha}(t_n)$, $\xi^{*\alpha}(t_n)$, and $\phi^{\beta}(t_n)$, $\phi^{*\beta}(t_n)$, $\alpha = 1, ..., n'$, $\beta = 1, ..., n''$, $j \in \mathbb{Z}$, the states of which are $\Xi(t_n) = (\xi_1(t_n),\ \xi_2(t_n))$ and $\Phi(t_n) = (\phi_1(t_n),\ \phi_2(t_n))$, where ξ and ϕ denote useful collective variables.

V' (V'') is the direct sum of two copies of a Hilbert space $\mathcal{H}$ and we will indicate its basis as $\{(|\alpha_1\rangle,\ |\alpha_2\rangle)\}$ ($\{(|\beta_1\rangle,\ |\beta_2\rangle)\}$), and ξ^{α} ($\phi^{*\beta}$) are the coefficient of vectors in $\mathcal{H}$ so that a vector in V' (V'') has coefficient $(\xi^{\alpha_1},\ \xi'^{\alpha_2})$ $((\phi^{\beta_1},\ \phi'^{\beta_2}))$.

As we have seen in the previous chapter, we can consider them as Hilbert space vectors in the Hilbert spaces V' and V'', respectively, with the inner product defined in eq.(3.42). Next, we consider the Hilbert space V that is the tensor product between V' and V''. Saying that V is the tensor product of V' and V'' means that we can take as a basis for

V: $\{(|\alpha_1\rangle,\ |\alpha_2\rangle) \otimes (|\beta_1\rangle,\ |\beta_2\rangle)\} = \{(|\alpha_1\rangle|\beta_1\rangle,\ |\alpha_1\rangle|\beta_2\rangle,\ |\alpha_2\rangle|\beta_1\rangle,\ |\alpha_2\rangle|\beta_2\rangle)\}$. A generic state of V can be written using the coefficients $(\psi_1^{\alpha_1\beta_1}, {\psi_2}^{\alpha_1\beta_2}, {\psi_3}^{\alpha_2\beta_1}, {\psi_4}^{\alpha_2\beta_2})$.

We use the following convention for the tensorial product between two matrices, one of them $n \times m$ ($\{a_{ij}\}$) and the other one $n' \times m'$ ($\{b_{kl}\}$):

$$\begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & \dots & b_{1m'} \\ \vdots & \ddots & \vdots \\ b_{n'1} & \dots & b_{n'm'} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11} \begin{pmatrix} b_{11} & \dots & b_{1m'} \\ \vdots & \ddots & \vdots \\ b_{n'1} & \dots & b_{n'm'} \end{pmatrix} & \dots & a_{1m} \begin{pmatrix} b_{11} & \dots & b_{1m'} \\ \vdots & \ddots & \vdots \\ b_{n'1} & \dots & b_{n'm'} \end{pmatrix} \\ \vdots & \ddots & \vdots \\ a_{n1} \begin{pmatrix} b_{11} & \dots & b_{1m'} \\ \vdots & \ddots & \vdots \\ b_{n'1} & \dots & b_{n'm'} \end{pmatrix} & \dots & a_{nm} \begin{pmatrix} b_{11} & \dots & b_{1m'} \\ \vdots & \ddots & \vdots \\ b_{n'1} & \dots & b_{n'm'} \end{pmatrix} \end{pmatrix} . \tag{4.23}$$

And the formula above can be used also for the tensor product between vectors, choosing $m = 1$ and $m' = 1$.

Let us see what happens if we consider a factored state. Given a state in V' and one in V'', both taken at the same discrete time, say they are $\Xi(t_n) = (\xi_1(t_n),\ \xi_2(t_n))$ and $\Phi(t_n) = (\phi_1(t_n),\ \phi_2(t_n))$, respectively, with $\xi_2(t_n) = \xi_1(t_n - l)$ and $\phi_2(t_n) = \phi_1(t_n - l)$, a factored state is written as $\Psi_{fac}(t_n) = \Xi(t_n) \otimes \Phi(t_n)$. We have that the coefficients of $\Xi(t_n)$ are $\{(\xi_1^{\alpha_1}(t_n),\ \xi_2^{\alpha_2}(t_n))\}$ and those of $\Phi(t_n)$ are $\{(\phi_1^{\beta_1}(t_n),\ \phi_2^{\beta_2}(t_n))\}$, and we can write the coefficients of $\Psi_{fac}(t_n)$ as:

$$\{(\xi_1^{\alpha_1}(t_n)\phi_1^{\beta_1}(t_n),\ \xi_1^{\alpha_1}(t_n)\phi_2^{\beta_2}(t_n),\ \xi_2^{\alpha_2}(t_n)\phi_1^{\beta_1}(t_n),\ \xi_2^{\alpha_2}(t_n)\phi_2^{\beta_2}(t_n))\}\ .$$

This gives us an hint how to characterize a non-factored state. It seems to be straightforward to characterize each coefficient of the non-factored state in the following way:

$$\Psi(t_n) = \{(\psi_1^{\alpha_1\beta_1}(t_n),\ {\psi_2}^{\alpha_1\beta_2}(t_n),\ {\psi_3}^{\alpha_2\beta_1}(t_n),\ {\psi_4}^{\alpha_2\beta_2}(t_n))\} \tag{4.24}$$

If we had considered the states $\Xi'(t_n) = (\xi_+(t_n),\ \xi_-(t_n))$ and $\Phi'(t_n) = (\phi_+(t_n),\ \phi_-(t_n))$, a factored state would have been written as $\Psi'_{fac}(t_n) = \Xi'(t_n) \otimes \Phi'(t_n)$. The coefficients of $\Xi'(t_n)$ are $\{(\xi_+^{\alpha_1}(t_n),\ \xi_-^{\alpha_2}(t_n))\}$ and those of $\Phi'(t_n)$ are $\{(\phi_+^{\beta_1}(t_n),\ \phi_-^{\beta_2}(t_n))\}$, so we can write those of $\Psi'_{fac}(t_n)$ as:

$$\{(\xi_+^{\alpha_1}(t_n)\phi_+^{\beta_1}(t_n),\ \xi_+^{\alpha_1}(t_n)\phi_-^{\beta_2}(t_n),\ \xi_-^{\alpha_2}(t_n)\phi_+^{\beta_1}(t_n),\ \xi_-^{\alpha_2}(t_n)\phi_-^{\beta_2}(t_n))\}\ .$$

Also in this case, this gives us an hint how to write a general non-factored state. It is:

$$\Psi'(t_n) = \{(\psi_{++}^{\alpha_1\beta_1}(t_n),\ \psi_{+-}{}^{\alpha_1\beta_2}(t_n),\ \psi_{-+}{}^{\alpha_2\beta_1}(t_n),\ \psi_{--}{}^{\alpha_2\beta_2}(t_n))\}\ . \tag{4.25}$$

From now on, for simplicity, we will consider only the state $\Psi(t_n)$, but it is straightforward to adapt our calculations for the state $\Psi'(t_n)$, paying attention to the fact that in this case we have a different form of the observables and of the time evolution operator (which we do not show here, since it is quite difficult to evaluate).

In Chapter (3), we built a C*-algebraic structure for the Hermitean operators and in particular for the observables $\hat{\mathbf{O}}^{\mathbf{G}}$. This will now be useful to understand what is the generic form of an observable and an operator acting on the space of states V.

First we consider Hermitean operators and observables that act only on one of the two Hilbert spaces V' or V'', leaving the other unchanged. These are of the form, respectively: $\hat{\mathbf{G}}' \otimes \hat{\mathbb{I}}''$, and $\hat{\mathbb{I}}' \otimes \hat{\mathbf{G}}''$ for the operators, and $\hat{\mathbf{O}}^{\mathbf{G}'} \otimes \hat{\mathbf{O}}^{\mathbb{I}''}$, $\hat{\mathbf{O}}^{\mathbb{I}'} \otimes \hat{\mathbf{O}}^{\mathbf{G}'}$ for the observables.

We will see shortly that the Hamiltonian that keeps a factored state factored will not be of the kind $\hat{\mathbf{H}}' \otimes \hat{\mathbb{I}}'' + \hat{\mathbb{I}}' \otimes \hat{\mathbf{H}}''$, where $\hat{\mathbf{H}}'$ acts on V' and $\hat{\mathbf{H}}''$ on V''. This is due to the fact that the "time derivative" we used is not a proper derivation (it does not obey the Leibniz rule), which causes some additional complication here.

A generic operator acting on V will not be of the kind described above, but will involve the tensor product between operators that act on the two Hilbert spaces V' and V''. We can consider an operator that mixes states of V' and V'' written as $\hat{\mathbf{G}}' \otimes \hat{\mathbf{G}}''$, then a generic operator acting on V will be a sum of operators of this kind.

Finally, we can write a generic observable associated with $\hat{\mathbf{G}}$ acting on V as an antidiagonal block matrix:

$$\hat{\mathbf{O}}^{\mathbf{G}} = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & \hat{G} \\ 0 & 0 & \hat{G} & 0 \\ 0 & \hat{G} & 0 & 0 \\ \hat{G} & 0 & 0 & 0 \end{pmatrix}\ . \tag{4.26}$$

This should be compared with the corresponding observable for a single HCA:

$$\hat{\mathbf{O}}^{\mathbf{G}} = \frac{1}{2}\begin{pmatrix} 0 & \hat{G} \\ \hat{G} & 0 \end{pmatrix} . \tag{4.27}$$

4.3.3 Updating equations on V

Next we want to write down updating equations for a factored state $\Psi_{fac}(t_n) \in V$. As we have seen in previous chapters we can write the updating equations for the two separate systems in the following way:

$$\begin{aligned} \Xi(t_n + l) &= \left[\Sigma_1' - i\left(\hat{\mathbf{I}}' + \Sigma_3'\right)\mathbf{H}'\right]\Xi(t_n) , \\ \Phi(t_n + l) &= \left[\Sigma_1'' - i\left(\hat{\mathbf{I}}'' + \Sigma_3''\right)\mathbf{H}''\right]\Phi(t_n) , \end{aligned} \tag{4.28}$$

where the two operator acting, respectively, on $\Xi(t_n)$ and $\Phi(t_n)$ are the time translation operators for one-time-step of the two systems; we can call them $\hat{\mathbf{T}}_1'$ and $\hat{\mathbf{T}}_1''$.

From these and the generic solutions for the systems, eq.(3.39), we get here solutions, given the two initial condition $\Xi(l)$ and $\Phi(l)$:

$$\begin{aligned} \Xi(t_n + l) &= \hat{\mathbf{T}}_n'\Xi(l) , \\ \Phi(t_n + l) &= \hat{\mathbf{T}}_n''\Phi(l) , \end{aligned} \tag{4.29}$$

where we have:

$$\begin{aligned} \hat{\mathbf{T}}_n' &= i^n \begin{pmatrix} -U_n(-cl\hat{H}') & iU_{n-1}(-cl\hat{H}') \\ iU_{n-1}(-cl\hat{H}') & U_{n-2}(-cl\hat{H}') \end{pmatrix} , \\ \hat{\mathbf{T}}_n'' &= i^n \begin{pmatrix} -U_n(-cl\hat{H}'') & iU_{n-1}(-cl\hat{H}'') \\ iU_{n-1}(-cl\hat{H}'') & U_{n-2}(-cl\hat{H}'') \end{pmatrix} , \end{aligned} \tag{4.30}$$

where $U_n(x)$ are the Chebyshev polynomials of the second kind. Now it is easy to find the updating equation for the factored state $\Psi_{fac}(t_n) = \Xi(t_n) \otimes \Phi(t_n)$. It is sufficient to take the tensor product between the two equations (4.28), which gives:

$$\Psi_{fac}(t_n + l) = \hat{\mathbf{T}}_1' \otimes \hat{\mathbf{T}}_1''\Psi_{fac}(t_n) , \tag{4.31}$$

where we have:

$$\Psi_{fac}(t_n) = \begin{pmatrix} \xi_1(t_n) \otimes \phi_1(t_n) \\ \xi_1(t_n) \otimes \phi_2(t_n) \\ \xi_2(t_n) \otimes \phi_1(t_n) \\ \xi_2(t_n) \otimes \phi_2(t_n) \end{pmatrix} \tag{4.32}$$

The solutions of (4.31), given $\Psi_{fac}(l)$ as initial condition are:

$$\Psi_{fac}(t_n) = \hat{\mathbf{T}}'_n \otimes \hat{\mathbf{T}}''_n \Psi_{fac}(l) \; . \tag{4.33}$$

Let us take a look at the structure of $\hat{\mathbf{T}}'_1 \otimes \hat{\mathbf{T}}''_1$, in order to guess how to generalize it. First, remember the structure of $\hat{\mathbf{T}}'_1$ and $\hat{\mathbf{T}}''_1$ that is:

$$\begin{aligned} \hat{\mathbf{T}}'_1 &= \begin{pmatrix} -2icl\hat{H}' & \hat{\mathbb{I}} \\ \hat{\mathbb{I}} & 0 \end{pmatrix} \; , \\ \hat{\mathbf{T}}''_1 &= \begin{pmatrix} -2icl\hat{H}'' & \hat{\mathbb{I}} \\ \hat{\mathbb{I}} & 0 \end{pmatrix} \end{aligned} \tag{4.34}$$

Then it is straightforward to obtain $\hat{\mathbf{T}}'_1 \otimes \hat{\mathbf{T}}''_1$, according to eq.(4.23):

$$\hat{\mathbf{T}}'_1 \otimes \hat{\mathbf{T}}''_1 = \begin{pmatrix} -4c^2l^2\hat{H}' \otimes \hat{H}'' & -2icl\hat{H}' \otimes \hat{\mathbb{I}} & -2icl\hat{\mathbb{I}} \otimes \hat{H}'' & \hat{\mathbb{I}} \otimes \hat{\mathbb{I}} \\ -2icl\hat{H}' \otimes \hat{\mathbb{I}} & 0 & \hat{\mathbb{I}} \otimes \hat{\mathbb{I}} & 0 \\ -2icl\hat{\mathbb{I}} \otimes \hat{H}'' & \hat{\mathbb{I}} \otimes \hat{\mathbb{I}} & 0 & 0 \\ \hat{\mathbb{I}} \otimes \hat{\mathbb{I}} & 0 & 0 & 0 \end{pmatrix} \; . \tag{4.35}$$

Note that the first term can be rewritten as $-4c^2l^2\hat{H}'\otimes\hat{H}'' = [-2icl\hat{H}'\otimes\hat{\mathbb{I}}][(-2icl)\hat{\mathbb{I}}\otimes\hat{H}'']$, where we are taking the scalar product between the two terms in square brackets. So it is the product between the second and the third terms in the first row (or column).

In a similar way, we can write down the structure of the operator $\hat{\mathbf{T}}'_n \otimes \hat{\mathbf{T}}''_n$. We start writing the two operators $\hat{\mathbf{T}}'_n$ and $\hat{\mathbf{T}}''_n$ that are:

$$\hat{\mathbf{T}}'_n = i^n \begin{pmatrix} -U_n(-cl\hat{H}') & iU_{n-1}(-cl\hat{H}') \\ iU_{n-1}(-cl\hat{H}') & U_{n-2}(-cl\hat{H}') \end{pmatrix} ,$$

$$\hat{\mathbf{T}}''_n = i^n \begin{pmatrix} -U_n(-cl\hat{H}'') & iU_{n-1}(-cl\hat{H}'') \\ iU_{n-1}(-cl\hat{H}'') & U_{n-2}(-cl\hat{H}'') \end{pmatrix} . \tag{4.36}$$

From now on, to unburden the notation we will write U'_n instead of $U_n(-cl\hat{H}')$ and U''_n instead of $U_n(-cl\hat{H}'')$. Then the operator $\hat{\mathbf{T}}'_n \otimes \hat{\mathbf{T}}''_n$ will be:

$$\hat{\mathbf{T}}'_n \otimes \hat{\mathbf{T}}''_n = i^{2n} \begin{pmatrix} U'_n \otimes U''_n & -iU'_n \otimes U''_{n-1} & -iU'_{n-1} \otimes U''_n & -U'_{n-1} \otimes U''_{n-1} \\ -iU'_n \otimes U''_{n-1} & -U'_n \otimes U''_{n-2} & -U'_{n-1} \otimes U''_{n-1} & iU'_{n-1} \otimes U''_{n-2} \\ -iU'_{n-1} \otimes U''_n & -U'_{n-1} \otimes U''_{n-1} & -U'_{n-2} \otimes U''_n & iU'_{n-2} \otimes U''_{n-1} \\ -U'_{n-1} \otimes U''_{n-1} & iU'_{n-1} \otimes U''_{n-2} & iU'_{n-2} \otimes U''_{n-1} & U'_{n-2} \otimes U''_{n-2} \end{pmatrix} . \tag{4.37}$$

To have a clearer idea of what is going on, we should rewrite the updating equations and the solutions, now for the vector components of $\Psi(t_n)$.

The updating equations are, after some rearrangement:

$$\begin{aligned} D[\xi_1(t_n) \otimes \phi_1(t_n)] &= -ic[\hat{H}'\xi_1(t_n) \otimes (-ilc\hat{H}''\phi_1(t_n) + \phi_2(t_n))] + \\ & -ic[(-ilc\hat{H}'\xi_1(t_n) + \xi_2(t_n)) \otimes \hat{H}''\phi_1(t_n)] , \\ [D\xi_1(t_n)] \otimes \phi_2(t_n + l) &= -ic[\hat{H}'\xi_1(t_n) \otimes \phi_2(t_n + l)] , \\ \xi_2(t_n + l) \otimes [D\phi_1(t_n)] &= -ic[\xi_1(t_n) \otimes \hat{H}''\phi_1(t_n)] , \\ \xi_2(t_n + l) \otimes \phi_2(t_n + l) &= \xi_1(t_n) \otimes \phi_1(t_n) , \end{aligned} \tag{4.38}$$

where as usual $D[O(t_n)] = (1/2l)(O(t_n + l) - O(t_n - l))$. Note that the first equation implies the secon and third ones, and the last one is a trivial identity. We can rewrite the first equation in terms of just ϕ_1 and ξ_1 in the following way:

$$\begin{aligned} &D[\xi_1(t_n) \otimes \phi_1(t_n)] = \\ -ic &\left\{ [\hat{H}'\xi_1(t_n) \otimes \tfrac{\phi_1(t_n+l)+\phi_1(t_n-l)}{2}] + [\tfrac{\xi_1(t_n+l)+\xi_1(t_n-l)}{2} \otimes \hat{H}''\phi_1(t_n)] \right\} , \end{aligned} \tag{4.39}$$

and because:

$$\begin{aligned} & D[\xi_1(t_n)\otimes\phi_1(t_n)] = \\ D[\xi_1(t_n)]\otimes\tfrac{\phi_1(t_n+l)+\phi_1(t_n-l)}{2} & + \tfrac{\xi_1(t_n+l)+\xi_1(t_n-l)}{2}\otimes D[\phi_1(t_n)]\ , \end{aligned} \tag{4.40}$$

we finally obtain:

$$\begin{aligned} & \Big\{D[\xi_1(t_n)] - ic\hat{H}'\xi_1(t_n)\Big\}\otimes\tfrac{\phi_1(t_n+l)+\phi_1(t_n-l)}{2} \ + \\ & \tfrac{\xi_1(t_n+l)+\xi_1(t_n-l)}{2}\otimes\Big\{D[\phi_1(t_n)] - ic\hat{H}''\phi_1(t_n)\Big\} = 0\ . \end{aligned} \tag{4.41}$$

In this form it is easy to see that this equation is solved by combining the two solutions of the second and third equations of (4.38), which are the solutions of the equations of the two separate systems.

Next we take a look at the explicit form of the solutions. After some rearrangement, they are given by:

$$\begin{aligned} \xi_1(t_n+l)\otimes\phi_1(t_n+l) &= [i^{n+1}(U'_{n-1}\xi_2(l)) + iU'_n\xi_1(l)]\otimes[i^{n+1}(U''_{n-1}\phi_2(l)+iU''_n\phi_1(l))]\ , \\ \xi_1(t_n+l)\otimes\phi_2(t_n+l) &= [i^{n+1}(U'_{n-1}\xi_2(l)) + iU'_n\xi_1(l)]\otimes[i^{n}(U''_{n-2}\phi_2(l)+iU''_{n-1}\phi_1(l))]\ , \\ \xi_2(t_n+l)\otimes\phi_1(t_n+l) &= [i^{n}(U'_{n-2}\xi_2(l)) + iU'_{n-1}\xi_1(l)]\otimes[i^{n+1}(U''_{n-1}\phi_2(l)+iU''_n\phi_1(l))]\ , \\ \xi_2(t_n+l)\otimes\phi_2(t_n+l) &= [i^{n}(U'_{n-2}\xi_2(l)) + iU'_{n-1}\xi_1(l)]\otimes[i^{n}(U''_{n-2}\phi_2(l)+iU''_{n-1}\phi_1(l))]\ , \end{aligned} \tag{4.42}$$

and if we substitute $\xi_2(t_n)$ and $\phi_2(t_n)$, respectively, with $\xi_1(t_n-l)$ and $\phi_1(t_n-l)$, we can easily see that these are compatible with the solutions of the two separate systems (see eq.(2.49)).

In this way, we have arrived at a consistent set of equations of motion for composite HCAs. This will be confirmed in the following sections, last not least, after introducing interaction among the subsystems.

4.3.4 Conservation laws

Here we want to explore the conservation laws for the composite system. Because the equations of motion for the two subsystems are separable, we can use **Theorem A'** for the two systems: Given an observable $\mathbf{\hat{O}^{G'}}$ for the first ($\mathbf{\hat{O}^{G''}}$ for the second) system we have that it is conserved, if and only if $\hat{G}'$ ($\hat{G}''$) commutes with $\hat{H}'$ ($\hat{H}''$).

As we have seen in section (4.3.2), the observables $\mathbf{\hat{O}^{G}}$ for the composite system are of the kind of eq.(4.26). We apply **Theorem A'**, in the form of Section (3.5.1), to these observables. Note that $\mathbf{\hat{O}^{G}} = \mathbf{\hat{O}^{G'}} \otimes \mathbf{\hat{O}^{G''}}$ is a sum of tensor products of the observables of the two single systems.

Following what has been done in the case of a single system, we obtain:

Theorem A" If $\hat{G}$ commutes both with $\hat{H}' \otimes \hat{\mathbb{I}}$ and $\hat{\mathbb{I}} \otimes \hat{H}''$ then the value of the observable $\mathbf{\hat{O}^{G}}$ evaluated w.r.t. a state $\Psi(t_n)$ is conserved.

*P*roof. To prove Theorem A" it is sufficient to show that if $\hat{G}$ commutes with both Hamiltonians then $\mathbf{\hat{T}}_1^\dagger \mathbf{\hat{O}^{G}} \mathbf{\hat{T}}_1 = \mathbf{\hat{O}^{G}}$. We have that:

$$\mathbf{\hat{T}}_1^\dagger \mathbf{\hat{O}^{G}} \mathbf{\hat{T}}_1 =$$

$$\frac{1}{4}\begin{pmatrix} -4c^2l^2\hat{M} & -2icl[\hat{G}, \hat{H}' \otimes \hat{\mathbb{I}}] & -2icl[\hat{G}, \hat{\mathbb{I}} \otimes \hat{H}''] & \hat{G} \\ -2icl[\hat{G}, \hat{H}' \otimes \hat{\mathbb{I}}] & 0 & \hat{G} & 0 \\ -2icl[\hat{G}, \hat{\mathbb{I}} \otimes \hat{H}''] & \hat{G} & 0 & 0 \\ \hat{G} & 0 & 0 & 0 \end{pmatrix}, \tag{4.43}$$

with:

$$\hat{M} = [\hat{H}' \otimes \hat{H}'']\hat{G} - [\hat{H}' \otimes \hat{\mathbb{I}}]\hat{G}[\hat{\mathbb{I}} \otimes \hat{H}''] - [\hat{\mathbb{I}} \otimes \hat{H}'']\hat{G}[\hat{H}' \otimes \hat{\mathbb{I}}] + \hat{G}[\hat{H}' \otimes \hat{H}''] \,. \tag{4.44}$$

Because $[\hat{H}' \otimes \hat{\mathbb{I}}, \hat{\mathbb{I}} \otimes \hat{H}''] = 0$, and if $\hat{G}$ commutes with both $\hat{H}' \otimes \hat{\mathbb{I}}$ and $\hat{\mathbb{I}} \otimes \hat{H}''$, we obtain $\hat{M} = 0$ and:

$$\mathbf{\hat{T}}_1^\dagger \mathbf{\hat{O}^{G}} \mathbf{\hat{T}}_1 = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & \hat{G} \\ 0 & 0 & \hat{G} & 0 \\ 0 & \hat{G} & 0 & 0 \\ \hat{G} & 0 & 0 & 0 \end{pmatrix} = \mathbf{\hat{O}^{G}} \,. \qquad \square \tag{4.45}$$

4.3.5 Continuum limit

As has been done for the case of single systems, we want to explore the continuum limit $l \to 0$, $t_n = t$, in the present case of composite systems.

We have seen that a factored state for the composite system is:

$$\Psi_{fac}(t_n) = \begin{pmatrix} \xi_1(t_n) \otimes \phi_1(t_n) \\ \xi_1(t_n) \otimes \phi_2(t_n) \\ \xi_2(t_n) \otimes \phi_1(t_n) \\ \xi_2(t_n) \otimes \phi_2(t_n) \end{pmatrix} . \tag{4.46}$$

Therefore, in the limit we get:

$$\lim_{l\to 0} \Psi_{fac}(t_n) = \Psi_{fac}(t) = \begin{pmatrix} \xi_1(t) \otimes \phi_1(t) \\ \xi_1(t) \otimes \phi_2(t) \\ \xi_2(t) \otimes \phi_1(t) \\ \xi_2(t) \otimes \phi_2(t) \end{pmatrix} . \tag{4.47}$$

The only difference between eq.(4.46) and eq.(4.47) is the substitution of the discrete time t_n with the continuum one t.

Note that because of the definition of $\xi_2(t_n) = \xi_1(t_n - l)$ (and analogously for that of $\phi_2(t_n) = \phi_1(t_n - l)$), in the continuum limit we obtain $\xi_2(t) = \xi_1(t)$ and $\phi_2(t) = \phi_1(t)$. Thus, the information we get from the state $\psi(t)$ is redundant; it is sufficient to know the first component.

Now let us take a look at the limit of the updating equations, recalling that $\xi_2(t_n) = \xi_1(t_n - l)$ and $\phi_2(t_n) = \phi_1(t_n - l)$:

$$\begin{gathered} D[\xi_1(t) \otimes \phi_1(t)] = -ic[\hat{H}'\xi_1(t) \otimes \phi_1(t)] + \\ -ic[\xi_1(t) \otimes \hat{H}''\phi_1(t)] \, , \\ [D\xi_1(t)] \otimes \phi_1(t) = -ic[\hat{H}'\xi_1(t) \otimes \phi_1(t)] \, , \\ \xi_1(t) \otimes [D\phi_1(t)] = -ic[\xi_1(t) \otimes \hat{H}''\phi_1(t)] \, , \\ \xi_2(t) \otimes \phi_2(t) = \xi_1(t) \otimes \phi_1(t) \, , \end{gathered} \tag{4.48}$$

where, in the limit, $D\xi_1(t) = \lim_{l\to 0}(\xi_1(t+l) - \xi_1(t-l))/(2l) = \lim_{l\to 0}(\xi_1(t+l) - \xi_2(t))/(2l)$,
$D\phi_1(t) = \lim_{l\to 0}(\phi_1(t+l) - \phi_1(t-l))/(2l) = \lim_{l\to 0}(\phi_1(t+l) - \phi_2(t))/(2l)$, and
$D[\xi_1(t)\otimes\phi_1(t)] = \lim_{l\to 0}(\xi_1(t+l)\otimes\phi_1(t+l) - \xi_1(t-l)\otimes\phi_1(t-l))/(2l) =$
$\lim_{l\to 0}(\xi_1(t+l)\otimes\phi_1(t+l) - \xi_2(t)\otimes\phi_2(t))/(2l)$. So D is a symmetric time derivative.

We can see that the eqs.(4.48) are still redundant and that the whole information is represented by the first one: It is equal to the Schrödinger equation in the case of two non-interacting systems.

Let us examine the generic solutions in this limit. If we consider the four eqs.(4.38), we find that they all go into one continuous equation:

$$\xi_1(t)\otimes\phi_1(t) = e^{-ic\hat{H}'t}\xi_1(0)\otimes e^{-ic\hat{H}''t}\phi_1(0)\ , \tag{4.49}$$

where the unitary operator with $\hat{H}'$ acts only on $\xi_1(0)$, while the unitary operator with $\hat{H}''$ acts only on $\phi_1(0)$. In writing eq.(4.49), we neglected the first order correction in l to $\xi_1(l)\otimes\phi_1(l)$, the reason why we can do this is discussed at the end of Appendix B for the more general case of non-factored states.

Eq.(4.49) is formally equal to the solution for two non-interacting quantum systems.

Finally, we consider the conservation laws. We know that if $[\hat{G}, \hat{H}'\otimes\hat{\mathbb{I}}] = [\hat{G}, \hat{\mathbb{I}}\otimes\hat{H}''] = 0$, then $\hat{\mathbf{T}}_1^\dagger\hat{\mathbf{O}}^{\mathbf{G}}\hat{\mathbf{T}}_1 = \hat{\mathbf{O}}^{\mathbf{G}}$ and this implies that $\langle\Psi(t_n), \hat{\mathbf{O}}^{\mathbf{G}}\Psi(t_n)\rangle$ does not depend on n. So, in the limit, we have that $\langle\Psi(t), \hat{\mathbf{O}}^{\mathbf{G}}\Psi(t)\rangle$ does not depend on t. We can explicitly write $\langle\Psi(t), \hat{\mathbf{O}}^{\mathbf{G}}\Psi(t)\rangle$:

$$\langle\Psi(t), \hat{\mathbf{O}}^{\mathbf{G}}\Psi(t) = \langle\xi_1(t)\otimes\phi_1(t), \hat{G}\xi_1(t)\otimes\phi_1(t)\rangle\ . \tag{4.50}$$

Thus, if $[\hat{G}, \hat{H}'\otimes\hat{\mathbb{I}}] = [\hat{G}, \hat{\mathbb{I}}\otimes\hat{H}''] = 0$, then the value of $\langle\xi_1(t)\otimes\phi_1(t), \hat{G}\xi_1(t)\otimes\phi_1(t)\rangle$ is conserved in time, which is what happens in quantum mechanics.

Note that because, the whole information on the system in the limit is given by $\xi_1\otimes\phi_1$, we can characterize a state by only considering it as the state, and taking as observables the operators $\hat{G}$ (remember that these are Hermitean operators), instead of $\hat{\mathbf{O}^{\mathbf{G}}}$. The operators $\hat{G}$ form a C-*algebra (if we consider the usual product between operators, i.e.g. considering the usual matrix product in the given representation). Moreover, if we consider the identity instead of a generic $\hat{G}$, we can see that $\langle\xi_1(t)\otimes\phi_1(t), \hat{\mathbb{I}}\xi_1(t)\otimes\phi_1(t)\rangle > 0$, for all $\xi_1(t)\otimes\phi_1(t)$, and, since it is a constant of motion, we can renormalize the states such that $\langle \xi_1(t) \otimes \phi_1(t), \hat{\mathbb{I}}\xi_1(t) \otimes \phi_1(t)\rangle = 1$. We may then introduce introduce the usual probabilistic interpretation of "quantum states" together with the Born rule.

4.3.6 Introducing the interactions

In the previous sections, we have seen how to combine two non- interacting systems, including the updating equations and the conservation laws for the discrete time HCAs and, then we studied the continuum limit.

Here we try to find a more general time step operator that mixes the states and, thus, allows to describe interactions between the subsystems. Our aim is to find the more general form of $\hat{\mathbf{T}}_1$ that mixes the states of the two systems. Furthermore, we would like to identify the observables $\hat{\mathbf{G}}$ that satisfy **Theorem A"**.

Let us recall the form of the non interacting $\hat{\mathbf{T}}_1$:

$$\hat{\mathbf{T}}_1 = \begin{pmatrix} -4c^2l^2\hat{H}'\otimes\hat{H}'' & -2icl\hat{H}'\otimes\hat{\mathbb{I}} & -2icl\hat{\mathbb{I}}\otimes\hat{H}'' & \hat{\mathbb{I}}\otimes\hat{\mathbb{I}} \\ -2icl\hat{H}'\otimes\hat{\mathbb{I}} & 0 & \hat{\mathbb{I}}\otimes\hat{\mathbb{I}} & 0 \\ -2icl\hat{\mathbb{I}}\otimes\hat{H}'' & \hat{\mathbb{I}}\otimes\hat{\mathbb{I}} & 0 & 0 \\ \hat{\mathbb{I}}\otimes\hat{\mathbb{I}} & 0 & 0 & 0 \end{pmatrix}. \tag{4.51}$$

We already noticed that the first term is $[-2icl\hat{H}'\otimes\hat{\mathbb{I}}][(-2icl)\hat{\mathbb{I}}\otimes\hat{H}'']$. So as a first guess we can try to substitute two operators that mix the states for $\hat{H}'\otimes\hat{\mathbb{I}}$ and $\hat{\mathbb{I}}\otimes\hat{H}''$. Let us call the two operators $\hat{\mathbf{H}}_1$ and $\hat{\mathbf{H}}_2$ respectively. They act on the full $\otimes$-space. We would get for the interacting one-time step evolution operator $\hat{\mathbf{T}}_1^{\mathbf{int}}$, in the commuting case, $[\hat{\mathbf{H}}_1, \hat{\mathbf{H}}_2] = 0$:

$$\hat{\mathbf{T}}_1^{\mathbf{int}} = \begin{pmatrix} -4c^2l^2\frac{\hat{\mathbf{H}}_1\hat{\mathbf{H}}_2+\hat{\mathbf{H}}_2\hat{\mathbf{H}}_1}{2} & -2icl\hat{\mathbf{H}}_1 & -2icl\hat{\mathbf{H}}_2 & \hat{I} \\ -2icl\hat{\mathbf{H}}_1 & 0 & \hat{I} & 0 \\ -2icl\hat{\mathbf{H}}_2 & \hat{I} & 0 & 0 \\ \hat{I} & 0 & 0 & 0 \end{pmatrix}, \tag{4.52}$$

where $\hat{I} = \hat{\mathbb{I}}\otimes\hat{\mathbb{I}}$ is the identity operator acting on each component of the state $\Psi(t_n)$.

Now suppose that there exists an observable $\hat{\mathbf{O}}^{\mathbf{G}}$ such that $[\hat{G}, \hat{\mathbf{H}}_1] = [\hat{G}, \hat{\mathbf{H}}_2] = 0$. We want to see if **Theorem A"** is satisfied, that is if $\hat{\mathbf{T}}_1^{\mathbf{int}\dagger}\hat{\mathbf{O}}^{\mathbf{G}}\hat{\mathbf{T}}_1^{\mathbf{int}} = \hat{\mathbf{O}}^{\mathbf{G}}$.

It is indeed easy easy to show this:

$$\hat{\mathbf{T}}_1^{\mathbf{int}\dagger}\hat{\mathbf{O}}^{\mathbf{G}}\hat{\mathbf{T}}_1^{\mathbf{int}} =$$

$$\begin{pmatrix} -4c^2l^2\hat{M} & -2icl[\hat{G},\hat{\mathbf{H}}_1] & -2icl[\hat{G},\hat{\mathbf{H}}_2] & \hat{G} \\ -2icl[\hat{G},\hat{\mathbf{H}}_1] & 0 & \hat{G} & 0 \\ -2icl[\hat{G},\hat{\mathbf{H}}_2] & \hat{G} & 0 & 0 \\ \hat{G} & 0 & 0 & 0 \end{pmatrix}, \tag{4.53}$$

with $\hat{M} = \hat{\mathbf{H}}_1\hat{\mathbf{H}}_2\hat{G} - \hat{\mathbf{H}}_1\hat{G}\hat{\mathbf{H}}_2 - \hat{\mathbf{H}}_2\hat{G}\hat{\mathbf{H}}_1 + \hat{G}\hat{\mathbf{H}}_1\hat{\mathbf{H}}_2$. And, because the two Hamiltonians commute with $\hat{G}$, we get:

$$\hat{\mathbf{T}}_1^{\mathbf{int}\dagger}\hat{\mathbf{O}}^{\mathbf{G}}\hat{\mathbf{T}}_1^{\mathbf{int}} =$$

$$\begin{pmatrix} -4c^2l^2\hat{M} & 0 & 0 & \hat{G} \\ 0 & 0 & \hat{G} & 0 \\ 0 & \hat{G} & 0 & 0 \\ \hat{G} & 0 & 0 & 0 \end{pmatrix}, \tag{4.54}$$

with $\hat{M} = \hat{G}\hat{\mathbf{H}}_1\hat{\mathbf{H}}_2 - \hat{G}\hat{\mathbf{H}}_2\hat{\mathbf{H}}_1$.

So, if we use the following one-time-step evolution operator:

$$\hat{\mathbf{T}}_1^{\mathbf{int}} = \begin{pmatrix} -4c^2l^2\frac{\hat{\mathbf{H}}_1\hat{\mathbf{H}}_2+\hat{\mathbf{H}}_2\hat{\mathbf{H}}_1}{2} & -2icl\hat{\mathbf{H}}_1 & -2icl\hat{\mathbf{H}}_2 & \hat{I} \\ -2icl\hat{\mathbf{H}}_1 & 0 & \hat{I} & 0 \\ -2icl\hat{\mathbf{H}}_2 & \hat{I} & 0 & 0 \\ \hat{I} & 0 & 0 & 0 \end{pmatrix}, \tag{4.55}$$

we get $\hat{\mathbf{T}}_1^{\mathbf{int}\dagger}\hat{\mathbf{O}}^{\mathbf{G}}\hat{\mathbf{T}}_1^{\mathbf{int}} = \hat{\mathbf{O}}^{\mathbf{G}}$ and **Theorem A"** is valid.

In particular we can see that if we consider the operator that in the continuum limit will correspond to the energy, $\hat{G} = \hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2$, the corresponding observable, $\hat{\mathbf{O}}^{\mathbf{H_1+H_2}}$ is conserved, due to the fact that the two Hamiltonians commute.

Now, for a generic non-factored state, the more general updating equation can be proposed to be:

$$\Psi(t_n + l) = \hat{\mathbf{T}}_1^{\mathbf{int}}\Psi(t_n)\ , \tag{4.56}$$

with:

$$\Psi(t_n) = \begin{pmatrix} \psi_1(t_n) \\ \psi_2(t_n) \\ \psi_3(t_n) \\ \psi_4(t_n) \end{pmatrix} , \tag{4.57}$$

where the differences between the $\psi_i(t_n)$ are of order $O(l)$.

Next, we want to find the n-time-steps operator $\hat{\mathbf{T}}^{\mathbf{int}}_n$ in the commuting case discussed above.

It is straightforward that $\hat{\mathbf{T}}^{\mathbf{int}}_n = (\hat{\mathbf{T}}^{\mathbf{int}}_1)^n$ and after some calculation (presented in Appendix (B)) we find:

$$\hat{\mathbf{T}}^{\mathbf{int}}_n = i^{2n} \begin{pmatrix} U^1_n U^2_n & -iU^1_n U^2_{n-1} & -iU^1_{n-1} U^2_n & -U^1_{n-1} U^2_{n-1} \\ -iU^1_n U^2_{n-1} & -U^1_n U^2_{n-2} & -U^1_{n-1} U^2_{n-1} & iU^1_{n-1} U^2_{n-2} \\ -iU^1_{n-1} U^2_n & -U^1_{n-1} U^2_{n-1} & -U^1_{n-2} U^2_n & iU^1_{n-2} U^2_{n-1} \\ -U^1_{n-1} U^2_{n-1} & iU^1_{n-1} U^2_{n-2} & iU^1_{n-2} U^2_{n-1} & U^1_{n-2} U^2_{n-2} \end{pmatrix} , \tag{4.58}$$

where $U^{1,2}_n = U_n(-cl\hat{\mathbf{H}}_{1,2})$.

This result allow us to write down the solution for $\Psi(t_n)$, with initial conditions $\Psi(l)$, that is:

$$\begin{aligned}
\psi_1(t_n + l) &= i^{2n} U^1_n U^2_n \psi_1(l) - i^{2n+1} U^1_n U^2_{n-1} \psi_2(l) + \\
&-i^{2n+1} U^1_{n-1} U^2_n \psi_3(l) - i^{2n} U^1_{n-1} U^2_{n-1} \psi_4(l) , \\
\psi_2(t_n + l) &= -i^{2n+1} U^1_n U^2_{n-1} \psi_1(l) - i^{2n} U^1_n U^2_{n-2} \psi_2(l) + \\
&-i^{2n} U^1_{n-1} U^2_{n-1} \psi_3(l) + i^{2n+1} U^1_{n-1} U^2_{n-2} \psi_4(l) , \\
\psi_3(t_n + l) &= -i^{2n+1} U^1_{n-1} U^2_n \psi_1(l) - i^{2n} U^1_{n-1} U^2_{n-1} \psi_2(l) + \\
&-i^{2n} U^1_{n-2} U^2_n \psi_3(l) + i^{2n} U^1_{n-2} U^2_{n-1} \psi_4(l) , \\
\psi_4(t_n + l) &= -i^{2n} U^1_{n-1} U^2_{n-1} \psi_1(l) + i^{2n+1} U^1_n U^2_{n-1} \psi_2(l) + \\
&i^{2n+1} U^1_{n-2} U^2_{n-1} \psi_3(l) + i^{2n} U^1_{n-2} U^2_{n-2} \psi_4(l) .
\end{aligned} \tag{4.59}$$

The fact that we use $\Psi(l)$ instead of $\Psi(0)$, is due to the choice we made in building the state Ψ.

The case in wich the two Hamiltonians do not commute is quite different for what concerns the conservation of $\hat{\mathbf{O}}^{\mathbf{H_1+H_2}}$.

Following what has been done in the commuting case using the one-time-step evolution operator as in eq.(4.55), we get:

$$\hat{\mathbf{T}}_1^{\mathbf{int}\dagger}\hat{\mathbf{O}}^{\mathbf{G}}\hat{\mathbf{T}}_1^{\mathbf{int}} =$$

$$\begin{pmatrix} -4c^2l^2\hat{M}' & -2icl[\hat{G},\hat{\mathbf{H}}_1] & -2icl[\hat{G},\hat{\mathbf{H}}_2] & \hat{G} \\ -2icl[\hat{G},\hat{\mathbf{H}}_1] & 0 & \hat{G} & 0 \\ -2icl[\hat{G},\hat{\mathbf{H}}_2] & \hat{G} & 0 & 0 \\ \hat{G} & 0 & 0 & 0 \end{pmatrix}, \tag{4.60}$$

with $\hat{M}' = (1/2)(\hat{\mathbf{H}}_1\hat{\mathbf{H}}_2\hat{G} - 2\hat{\mathbf{H}}_1\hat{G}\hat{\mathbf{H}}_2 - 2\hat{\mathbf{H}}_2\hat{G}\hat{\mathbf{H}}_1 + \hat{G}\hat{\mathbf{H}}_1\hat{\mathbf{H}}_2 + \hat{\mathbf{H}}_2\hat{\mathbf{H}}_1\hat{G} + \hat{G}\hat{\mathbf{H}}_2\hat{\mathbf{H}}_1)$.

Then, if we consider $\hat{G} = \hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2$ we obtain:

$$\hat{\mathbf{T}}_1^{\mathbf{int}\dagger}\hat{\mathbf{O}}^{\mathbf{H_1+H_2}}\hat{\mathbf{T}}_1^{\mathbf{int}} =$$

$$\begin{pmatrix} -4c^2l^2\hat{M}'' & -2icl[\hat{\mathbf{H}}_2,\hat{\mathbf{H}}_1] & -2icl[\hat{\mathbf{H}}_1,\hat{\mathbf{H}}_2] & \hat{\mathbf{H}}_1+\hat{\mathbf{H}}_2 \\ -2icl[\hat{\mathbf{H}}_2,\hat{\mathbf{H}}_1] & 0 & \hat{\mathbf{H}}_1+\hat{\mathbf{H}}_2 & 0 \\ -2icl[\hat{\mathbf{H}}_1,\hat{\mathbf{H}}_2] & \hat{\mathbf{H}}_1+\hat{\mathbf{H}}_2 & 0 & 0 \\ \hat{\mathbf{H}}_1+\hat{\mathbf{H}}_2 & 0 & 0 & 0 \end{pmatrix}, \tag{4.61}$$

with $\hat{M}'' = 0$. Thus, since the two Hamiltonians do not commute we obtain $\hat{\mathbf{T}}_1^{\mathbf{int}\dagger}\hat{\mathbf{O}}^{\mathbf{H_1+H_2}}\hat{\mathbf{T}}_1^{\mathbf{int}} \neq \hat{\mathbf{O}}^{\mathbf{H_1+H_2}}$ and the observable corresponding to the energy is not conserved.

However, we will see that for both cases, the commuting one and the non-commuting one, the continuum limit of the solution for $\psi_1(t)$ $(= \psi_2(t) = \psi_3(t) = \psi_4(t))$ is $e^{-iHt}\psi_1(0)$, where $H = \hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2$. Thus, in the continuum limit, the time evolution is unitary and the energy conserved, as in quantum mechanics.

4.3.6.1 Continuum limit

Here we want to study the continuum limit for composite systems, when $[\hat{\mathbf{H}}_1, \hat{\mathbf{H}}_2] = 0$, recalling that the continuum limit is done for $l \to 0$ and $t_n = ln = t$. First, consider the initial condition $\Psi(l)$. A priori we have to keep it up to first order in l, however, as is shown for the general case of non-commuting Hamiltonians in the end of Appendix B, we can consider just its 0^{th} order. So we have:

$$\lim_{l\to 0}\Psi(l) = \begin{pmatrix}\psi_1(0)\\ \psi_1(0)\\ \psi_1(0)\\ \psi_1(0)\end{pmatrix} . \tag{4.62}$$

The next step is to see what happens to the updating equations. Taking the limit of eqs.(4.56), we get:

$$\begin{aligned}
\lim_{l\to 0}\tfrac{\psi_1(t_n+l)-\psi_4(t_n)}{2l} &= \lim_{l\to 0}\Big(-ic\hat{\mathbf{H}}_1\psi_2(t_n) - ic\hat{\mathbf{H}}_2\psi_3(t_n)\Big) \ ,\\
\lim_{l\to 0}\tfrac{\psi_2(t_n+l)-\psi_3(t_n)}{2l} &= \lim_{l\to 0}\Big(-ic\hat{\mathbf{H}}_1\psi_1(t_n)\Big) \ ,\\
\lim_{l\to 0}\tfrac{\psi_3(t_n+l)-\psi_2(t_n)}{2l} &= \lim_{l\to 0}\Big(-ic\hat{\mathbf{H}}_2\psi_1(t)\Big) \ ,\\
\lim_{l\to 0}\tfrac{\psi_4(t_n+l)-\psi_4(t_n-l)}{2l} &= -ic\hat{\mathbf{H}}_1\psi_2(t_n-l) - ic\hat{\mathbf{H}}_2\psi_3(t_n-l) \ .
\end{aligned} \tag{4.63}$$

To leading order, we can write $\psi_4(t_n) = \psi_1(t_n - l)$. In the second and the third equation, we can use the expansion to first order of, respectively, $\psi_3(t_n)$ and $\psi_2(t_n)$, which gives $\psi_3(t_n) \sim \psi_2(t_n-l) - ic\hat{\mathbf{H}}_2\psi_2(t_n-l)$ and $\psi_2(t_n) \sim \psi_3(t_n-l) - ic\hat{\mathbf{H}}_1\psi_3(t_n-l)$. Moreover, we can substitute the $\psi_i(t_n)$ on the r.h.s. of the first three eqs. of (4.63) with their 0^{th} order values, that is $\psi_1(t_n) \sim \psi_2(t_n) \sim \psi_3(t_n) \sim \psi_4(t_n)$. At the end, we get the equations:

$$\begin{aligned}
D\psi_1(t) &= -ic(\hat{\mathbf{H}}_1+\hat{\mathbf{H}}_2)\psi_1(t_n) \ ,\\
D\psi_2(t) &= -ic(\hat{\mathbf{H}}_1+\hat{\mathbf{H}}_2)\psi_2(t_n) \ ,\\
D\psi_3(t) &= -ic(\hat{\mathbf{H}}_1+\hat{\mathbf{H}}_2)\psi_3(t) \ ,\\
D\psi_4(t) &= -ic(\hat{\mathbf{H}}_1+\hat{\mathbf{H}}_2)\psi_4(t) \ .
\end{aligned} \tag{4.64}$$

So we got four identical equation with four identical initial condition, which means that the information given by $\Psi(t)$ is redundant and, as in the case of a single system, we can reduce the space of states to ψ_1 and the observables to Hermitean matrices $\hat{G}$.

Finally, just to check the consistency of the limit, we can explicitly evaluate $\lim_{l\to 0,n\to\infty} \hat{\mathbf{T}}_n^{\mathbf{int}}\Psi(l)$ (we will denote $\lim_{l\to 0,n\to\infty}$ by just $\lim_{l\to 0}$) to obtain:

$$\begin{aligned}
\psi_1(t) &= \lim_{l\to 0} i^{n+1}(iU_n^1 + U_{n-1}^1)i^{n+1}(iU_n^2 + U_{n-1}^2)\psi_1(0) = e^{-ic(\hat{\mathbf{H}}_1+\hat{\mathbf{H}}_2)t}\psi_1(0)\ , \\
\psi_2(t) &= \lim_{l\to 0} i^{n+1}(iU_n^1 + U_{n-1})i^{n}(iU_{n-1}^2 + iU_{n-2}^2)\psi_2(0) = e^{-ic(\hat{\mathbf{H}}_1+\hat{\mathbf{H}}_2)t}\psi_2(0)\ , \\
\psi_3(t) &= \lim_{l\to 0} i^{n}(iU_{n-1}^1 + iU_{n-2}^1)i^{n+1}(iU_n^2 + iU_{n-1})\psi_3(0) = e^{-ic(\hat{\mathbf{H}}_1+\hat{\mathbf{H}}_2)t}\psi_3(0)\ , \\
\psi_4(t) &= \lim_{l\to 0} i^{n}(iU_{n-1}^1 + iU_{n-2}^1)i^{n}(iU_{n-1}^2 + iU_{n-2}^2)\psi_4(0) = e^{-ic(\hat{\mathbf{H}}_1+\hat{\mathbf{H}}_2)t}\psi_4(0)\ ,
\end{aligned} \tag{4.65}$$

that are the solutions of eqs.(4.64).

Note that, even if the initial conditions $\psi_1(0) = \psi_2(0) = \psi_3(0) = \psi_4(0)$, are factored vectors, the solutions $\psi_1(t) = \psi_2(t) = \psi_3(t) = \psi_4(l)$ can be non-factored, due to the fact that the Hamiltonian contains interaction terms between the two subsystems.

4.4 The action of composite systems

We have written the updating equations for the composite HCA. Now we want to show that a generalization of the single system's action (3.66) can be written for the composite system. Indeed, we can see that we get the updating equations from the following action:

$$S := \sum_n \langle \Psi(t_n), \Sigma_1 \Psi(t_n + l)\rangle - \langle \Psi(t_n), \Sigma_1 \hat{\mathbf{T}}_1^{\mathbf{int}} \Psi(t_n)\rangle\ , \tag{4.66}$$

where $\Psi(t_n)$ is the state of the composite system and $\hat{\mathbf{T}}_1^{\mathbf{int}}$ is that of eq.(4.55). If we variate S w.r.t. $\Psi^\dagger(t_n)$, we get for $\Psi(t_n)$ the equation:

$$\Sigma_1 \Psi(t_n + l) = \Sigma_1 \hat{\mathbf{T}}_1^{\mathbf{int}} \Psi(t_n)\ , \tag{4.67}$$

that is equivalent to eq.(4.56), while, if we variate w.r.t. $\Psi(t_n)$, we get for $\Psi^\dagger(t_n)$:

$$\Psi^\dagger(t_n - l)\Sigma_1 = \Psi^\dagger(t_n)\Sigma_1 \hat{\mathbf{T}}_1^{\mathbf{int}}\ . \tag{4.68}$$

Now, because we can multiply both sides by Σ_1 from the right without changing the content of the equation, and because $\Sigma_1 \hat{\mathbf{T}}_1^{\mathbf{int}} \Sigma_1 = \hat{\mathbf{T}}_{-1}^{\mathbf{int}\dagger}$ we have:

$$\Psi^\dagger(t_n - l) = \Psi^\dagger(t_n)\hat{\mathbf{T}}_{-1}^{\mathbf{int}\dagger} \ , \tag{4.69}$$

which is the Hermitean conjugate of the backward updating equation for $\Psi(t_n)$. So we can use a formal action of kind (4.66) for both, single and composite systems, changing just the expressions for Ψ and $\hat{\mathbf{T}}_1^{\mathbf{int}}$ for the two different cases.

This concludes our construction of the formal description of single and bi-partite HCA systems. Apart from the technical developments, our main result is that the tensor product structure, which is the characteristic of composite quantum mechanical systems, can already be consistently implemented at the level of discrete Hamiltonian Cellular Automata. In the limit of vanishing discreteness scale ($l \to 0$), the quantum mechanical structure is fully recovered.

Chapter 5

Numerical studies

5.1 Introduction

In the previous chapters we have studied the behaviour and the formal description, in particular, of what we called Hamiltonian Cellular Automata (HCA), in order to understand the similarities and differences between them and quantum mechanical systems.

In Chapter (2), we introduced the single HCAs giving their action and updating equations. Then, we studied their solutions and conservation laws, noticing that in the continuum limit $l \to 0$, $t_n = ln = t$, the updating equations go into the Schrödinger equation, so the solutions and conservation laws for the HCAs become the same as those of quantum mechanics. We also noticed that, for l finite, the conservation laws are slightly different from the quantum mechanical ones, but we could still find parallels, since there is a one-to-one correspondence between them.

Starting from the considerations of Chapter (2), in Chapter (3), we tried to build a structure for the space of states and the observables of the HCA that should resemble that of quantum mechanics, introducing the notion of non-commuting C*-algebra for the observables. Before that, having noticed that the updating equations are of 2^{nd} order, we defined states that involved a doubled number of degrees of freedom w.r.t. those of quantum mechanics.

In Chapter 4, thanks to the structure of observables, we have been able to build a tensor product structure for composite HCA and we completed the study of Chapter (3) for it.

Here we want to complete our study of the single HCA looking for the behaviour of the solutions in the long-time limit with l finite, $n \to \infty$. Depending on the eigenvalues of the Hamiltonian, we will observe very different behaviour of the states of our HCA.

In this chapter, we abandon the summation convention for repeated greek indices.

In the graphics we will show, the variable n will be taken continuous, this is possible, because the Chebyshev polynomials of the second kind U_n can be generalized to real n. This becomes obvious for when their argument is less than 1, because we can use their trigonometric definition:

$$U_n[\cos(\theta)] = \frac{\sin[(n+1)\theta]}{\sin\theta} \;, \tag{5.1}$$

which is valid also for n real. Obviously in first place we will be interested only in integer values of n, but the figures are more readable, if we allow n to be real.

Moreover, if we use Shannon Theorem, as in [1], to continue our functions, the solutions for the system take the same form as in the discrete case simply as function of a real time n. In fact, the Shannon Theorem allows us to write $\sin(t) = \sum_n \sin t_n sinc[\pi(t/l - n)]$, where $t_n = nl$, and our solutions are sums of such sine functions.

5.2 The behaviour of the states

Recall that we have written a state of the HCA as $\Psi(t_n) = (\psi_1(t_n), \psi_2(t_n))$, or $\Psi'(t_n) = (\psi_+(t_n), \psi_-(t_n))$, with $\psi_-(t_n) = (1/2)(\psi_1(t_n) - \psi_2(t_n))$ of order $O(l)$ w.r.t. $\psi_+(t_n) = (1/2)(\psi_1(t_n) + \psi_2(t_n))$. We want to see what happens to the state $\Psi(t_n)$ or equivalently to the state $\Psi'(t_n)$ in the limit $n \to \infty$, l finite. So we need to study the behaviour in that limit of $\psi_1(t_n)$ (recalling that $\psi_2(t_n) = \psi_1(t_n - l)$) and that of $\psi_+(t_n)$ and $\psi_-(t_n)$, actually, we will not consider $\psi_-(t_n)$. In particular, we want to see if these Hilbert space vectors can have a dominant part, because in this case the behaviour of the HCA will be very different from that of the quantum mechanical counterpart. For this purpose, we will study the two normalized Hilbert space vectors:

$$\begin{aligned} \psi_1^{nor}(t_n) &= \frac{\psi_1(t_n)}{\psi_1^\dagger(t_n)\psi_1(t_n)} \;, \\ \psi_+^{nor}(t_n) &= \frac{\psi_+(t_n)}{\psi_+^\dagger(t_n)\psi_+(t_n)} \;. \end{aligned} \tag{5.2}$$

Studying this in the general case (for general initial conditions) is quite difficult, so we will study the simpler case in which $\psi_1(l) = \psi_2(l) = \psi_1(0)$ (that means $\psi_-(l) = 0$). In general, we can suppose that both $\psi_+(l)$ and $\psi_-(l)$ are different from zero.

Our simplified choice of initial conditions is just a question of simplicity of the numerical evaluation. Besides this, we are also making the hypotesis that l is very small, in many case smaller than the experimental limitations on the measurement of time intervals.

All we need now, are the explicit solutions for $\psi_1(t_n)$ and $\psi_+(t_n)$ for initial conditions $\psi_1(l) = \psi_2(l)$ or $\psi_+(l)$, $\psi_-(l) = 0$ which are given by:

$$\begin{aligned} \psi_1(t_n) &= i^n(U_{n-2} + iU_{n-1})\psi_1(l) \ , \\ \psi_+(t_n) &= i^n(U_{n-2} + iU_{n-1} + il\hat{H}U_{n-2})\psi_+(l) \ , \end{aligned} \tag{5.3}$$

where U_n is the n^{th} Chebyshev polynomial of the second kind, the omitted argument of which is $-l\hat{H}$.

The study will be easier for the Hamiltonian eigenstates orthonormal basis, which vectors will be denoted with ψ^α, where α labels the m eigenvalues $\{\epsilon_\alpha\}$, which we consider ordered such that $|\epsilon_1| > |\epsilon_2| > ... > |\epsilon_m|$. So we can write:

$$\begin{aligned} \psi_1(l) &= \textstyle\sum_{\alpha=1}^m b_\alpha \psi^\alpha \ , \\ \psi_+(l) &= \textstyle\sum_{\alpha=1}^m c_\alpha \psi^\alpha \ . \end{aligned} \tag{5.4}$$

With the help of this decomposition, we can write the solutions (5.3) as:

$$\begin{aligned} \psi_1(t_n) &= i^n \textstyle\sum_{\alpha=1}^m \Big(U_{n-2}(-l\epsilon_\alpha) + iU_{n-1}(-l\epsilon_\alpha)\Big) b_\alpha \psi^\alpha \ , \\ \psi_+(t_n) &= i^n \textstyle\sum_{\alpha=1}^m \Big(U_{n-2}(-l\epsilon_\alpha) + iU_{n-1}(-l\epsilon_\alpha) + il\epsilon_\alpha U_{n-2}(-l\epsilon_\alpha)\Big) c_\alpha \psi^\alpha \ , \end{aligned} \tag{5.5}$$

where now the Chebyshev polynomials have real arguments (instead of being polynomials of the Hermitean matrix $\hat{H}$). From now on, we will call $\rho_\alpha = -l\epsilon_\alpha$. Thus, the normalized vectors become:

$$\begin{aligned} \psi_1^{nor}(t_n) &= i^n \textstyle\sum_{\alpha=1}^m \frac{\Big(U_{n-2}(\rho_\alpha) + iU_{n-1}(\rho_\alpha)\Big) b_\alpha \psi^\alpha}{\sqrt{\sum_{\beta=1}^n b_\beta^* b_\beta |U_{n-2}(\rho_\beta) + iU_{n-1}(\rho_\beta)|^2}} \ , \\ \psi_+^{nor}(t_n) &= i^n \textstyle\sum_{\alpha=1}^m \frac{\Big(U_{n-2}(\rho_\alpha) + iU_{n-1}(\rho_\alpha) - i\rho_\alpha U_{n-2}(\rho_\alpha)\Big) c_\alpha \psi^\alpha}{\sqrt{\sum_{\beta=1}^n c_\beta^* c_\beta |U_{n-2}(\rho_\beta) + iU_{n-1}(\rho_\beta) - i\rho_\beta U_{n-2}(\rho_\beta)|^2}} \ , \end{aligned} \tag{5.6}$$

where we used the fact that $\langle \psi^\alpha, \psi^\beta \rangle = \delta_{\alpha\beta}$.

In to understand what happens to each component, we project the two vectors on ψ^α for each $\alpha = 1, ..., m$, obtaining m complex coefficients, then we take the absolute value of those coefficients denoting them as Z_1^α for ψ_1^{nor} and Z_+^α for ψ_+^{nor}. We get:

$$\begin{aligned} Z_1^\alpha(t_n) &= \frac{\left|U_{n-2}(\rho_\alpha)+iU_{n-1}(\rho_\alpha)\right|^2 b_\alpha^* b_\alpha}{\sum_{\beta=1}^m \left|U_{n-2}(\rho_\beta)+iU_{n-1}(\rho_\beta)\right|^2 b_\beta^* b_\beta} \,, \\ Z_+^\alpha(t_n) &= \frac{\left|U_{n-2}(\rho_\alpha)+iU_{n-1}(\rho_\alpha)-i\rho_\alpha U_{n-2}(\rho_\alpha)\right|^2 c_\alpha^* c_\alpha \psi^\alpha}{\sum_{\beta=1}^m \left|U_{n-2}(\rho_\beta)+iU_{n-1}(\rho_\beta)-i\rho_\beta U_{n-2}(\rho_\beta)\right|^2 c_\beta^* c_\beta} \,. \end{aligned} \tag{5.7}$$

$Z_1^\alpha(t_n)$ ($Z_+^\alpha(t_n)$) gives us the relative weight of the α component of the vector $\psi_1(t_n)$ ($\psi_+(t_n)$).

As we already said the eigenvalues of $\hat{H}$, and so the ρ_α, are ordered from the one with the biggest to the one with the smallest absolute value. Now we divide both the numerator and the denominator by $Z_1^1(t_n)$ obtaining:

$$\begin{aligned} Z_1^\alpha(t_n) &= \frac{\frac{\left|U_{n-2}(\rho_\alpha)+iU_{n-1}(\rho_\alpha)\right|^2 b_\alpha^* b_\alpha}{\left|U_{n-2}(\rho_1)+iU_{n-1}(\rho_1)\right|^2 b_1^* b_1}}{1+\frac{\sum_{\beta=2}^m \left|U_{n-2}(\rho_\beta)+iU_{n-1}(\rho_\beta)\right|^2 b_\beta^* b_\beta}{\left|U_{n-2}(\rho_1)+iU_{n-1}(\rho_1)\right|^2 b_1^* b_1}} \,, \\ Z_+^\alpha(t_n) &= \frac{\frac{\left|U_{n-2}(\rho_\alpha)+iU_{n-1}(\rho_\alpha)-i\rho_\alpha U_{n-2}(\rho_\alpha)\right|^2 c_\alpha^* c_\alpha}{\left|U_{n-2}(\rho_1)+iU_{n-1}(\rho_1)-i\rho_1 U_{n-2}(\rho_1)\right|^2 c_1^* c_1}}{1+\frac{\sum_{\beta=2}^m \left|U_{n-2}(\rho_\beta)+iU_{n-1}(\rho_\beta)-i\rho_\beta U_{n-2}(\rho_\beta)\right|^2 c_\beta^* c_\beta}{\left|U_{n-2}(\rho_1)+iU_{n-1}(\rho_1)-i\rho_1 U_{n-2}(\rho_1)\right|^2 c_1^* c_1}} \,. \end{aligned} \tag{5.8}$$

From the above expressions it is easy to see that their behaviour in time is determined by that of the quantities:

$$\begin{aligned} P_1^\alpha(n) &= \frac{\left|U_{n-2}(\rho_\alpha)+iU_{n-1}(\rho_\alpha)\right|^2 b_\alpha^* b_\alpha}{\left|U_{n-2}(\rho_1)+iU_{n-1}(\rho_1)\right|^2 b_1^* b_1} \,, \\ P_+^\alpha(n) &= \frac{\left|U_{n-2}(\rho_\alpha)+iU_{n-1}(\rho_\alpha)-i\rho_\alpha U_{n-2}(\rho_\alpha)\right|^2 c_\alpha^* c_\alpha}{\left|U_{n-2}(\rho_1)+iU_{n-1}(\rho_1)-i\rho_1 U_{n-2}(\rho_1)\right|^2 c_1^* c_1} \,. \end{aligned} \tag{5.9}$$

For a preliminary study we can neglect the ratio $|b_\alpha|^2/|b_1|^2$ for the first and the ratio $|c_\alpha|^2/|c_1|^2$ (because we are interested in the qualitative behaviour in time and we are neglecting just constant factors) for the second. Therefore, we consider simply the behaviour in time of the following two functions:

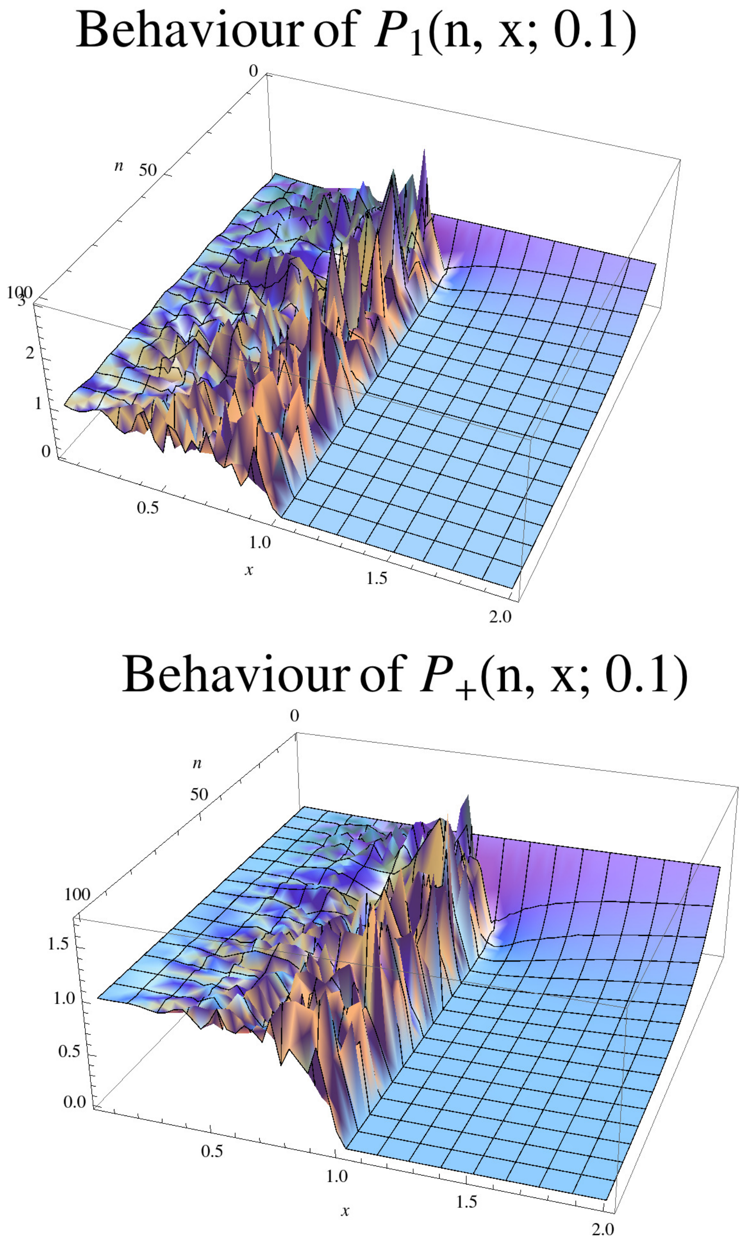
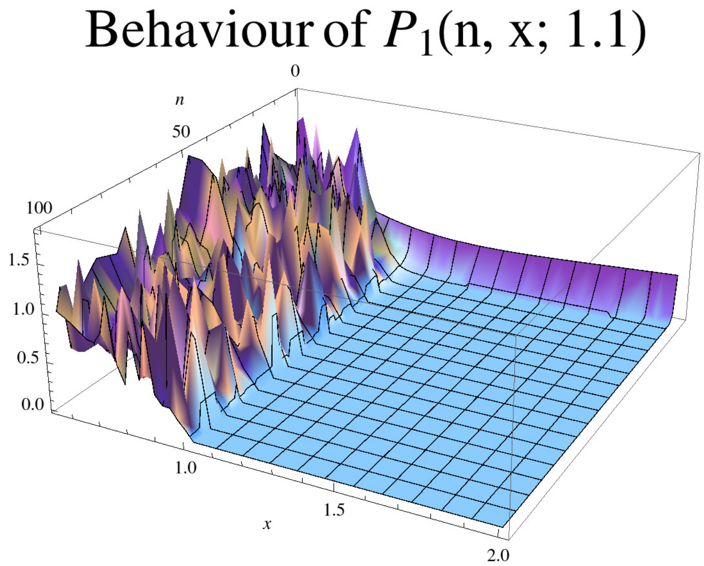


FIGURE 5.1: In this figure, we plotted $P_1(n, x; 0.1)$ in the upper figure and $P_+(n, x; 0.1)$ in the bottom one, in the region $0.05 < x < 2$. In both cases the behaviour changes drastically for $x \geq 1$.

$$
\begin{aligned}
P_1(n, \rho; \Delta) &= \frac{\left|U_{n-2}(\rho-\Delta)+iU_{n-1}(\rho-\Delta)\right|^2}{\left|U_{n-2}(\rho)+iU_{n-1}(\rho)\right|^2}\,, \\
P_+(n, \rho; \Delta) &= \frac{\left|U_{n-2}(\rho-\Delta)+iU_{n-1}(\rho-\Delta)-i(\rho-\Delta)U_{n-2}(\rho-\Delta)\right|^2}{\left|U_{n-2}(\rho)+iU_{n-1}(\rho)-i\rho U_{n-2}(\rho)\right|^2}\,,
\end{aligned}
\tag{5.10}
$$

where $\Delta > 0$. We are interested in the functions above in the case $|\rho - \Delta| < |\rho|$, that is the region $\rho > \Delta/2$.



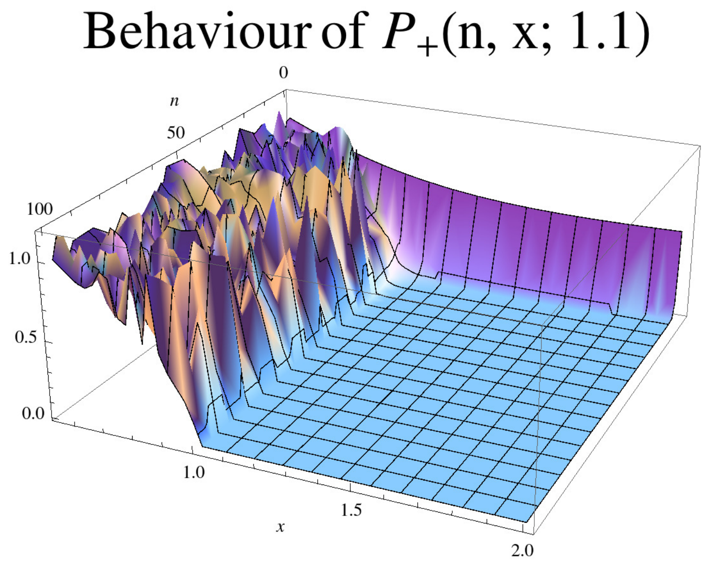


FIGURE 5.2: In this figure, we plotted $P_1(n, x; 1.1)$ in the upper figure and $P_+(n, x; 1.1)$ in the bottom one in the region $0.05 < x < 2$. In both cases the behaviour changes drastically for $x > 1$.

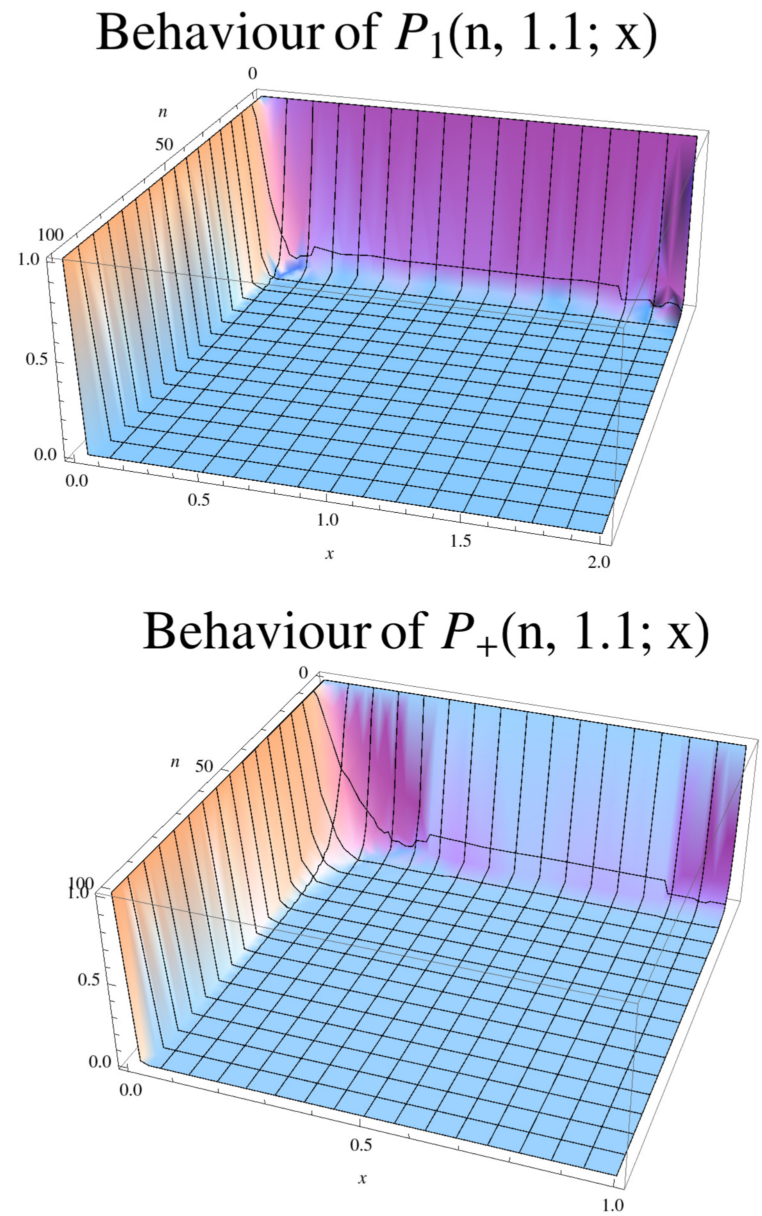


FIGURE 5.3: In this figure, we plotted $P_1(n, 1.1; x)$ in the upper figure and $P_+(n, 1.1; x)$ in the bottom one in the region $0 < x < 2$. In both cases $P \to 0$ when $n \to \infty$ for $x > 0$.

In fig. 5.1, we plotted $P_1(n, x, 0.1)$ and $P_+(n, x, 0.1)$, while in fig. 5.2 we plotted $P_1(n, x, 1.1)$ and $P_+(n, x, 1.1)$, then in fig. 5.3 we plotted $P_1(n, 1.1, x)$ and $P_+(n, 1.1, x)$, where x is the variable considered, i.e. ρ and Δ, respectively. From these figures we guess the behaviour of the normalized Hilbert space vectors for $|\rho_1| > 1$. In fact, we learn that for $n \to \infty$ all the $P_1^\alpha(t_n)$ and $P_+^\alpha(t_n)$ go to 0 except for $P_1^1(t_n)$ and $P_+^1(t_n)$. So we get that:

$$
\begin{aligned}
&P_1^1 \to 1 \;\text{ for }\; n \to \infty \;, \\
&P_+^1 \to 1 \;\text{ for }\; n \to \infty \;,
\end{aligned}
\tag{5.11}
$$

and for $\alpha \geq 2$:

$$
\begin{aligned}
&P_1^\alpha \to 0 \;\text{ for }\; n \to \infty \;, \\
&P_+^\alpha \to 0 \;\text{ for }\; n \to \infty \;,
\end{aligned}
\tag{5.12}
$$

which means that for both vectors, $\psi_1(t_n)$ and $\psi_+(t_n)$, survives only the component corresponding to the eigenvalue of the Hamiltonian with the biggest absolut value.

Moreover, even if we have not illustrated this, we find that in this situation, both $|\psi_1(t_n)|^2|$ and $|\psi_+(t_n)|^2$, grow indefinitely in time (linearly for $\rho_1 = 1$, and exponentially for $\rho_1 > 1$).

This is because, as we already pointed out, the norm is not conserved. Despite this somewhat unexpected behaviour, the quantity $\langle \Psi(t_n), \Sigma_1 \Psi(t_n) \rangle$ is a constant of motion, as we have demonstrated generally.

A more interesting situation seems to be that in which $\max(\{|\rho_\alpha|\}) < 1$. In this case, it is not so easy to understand the behaviour of the Hilbert space vectors from that of the functions shown in figs.5.1, 5.2, 5.3. Therefore, in the next section we will perform a numerical simulation for a sufficiently small state space based on a two-dimensional Hilbert space ($m = 2$).

To complete the present study, we should look also at $\psi_-(t_n)$. The problem is that if we try to repeat for it the procedure we have followed for $\psi_1(t_n)$ and $\psi_+(t_n)$ in the case $|\rho| < 1$, we will not get any useful information.

In fact, following what we have done before, we can define $P_-^\alpha(t_n)$ for each component and then study the function $P_-(n, \rho; \Delta)$.

This function is shown in fig. 5.4 and, as we can see, for $\rho \leq 1$ it has very high and isolated peaks that do not allow us to see simultaneously what happens in the rest of the graph. While for $\rho > 1$, its behaviour is the same as that of the other two Hilbert space vectors, as expected. This general behaviour (high isolated peaks) occur also if we consider smaller regions, due to the fact that the denominator can take values very near 0.

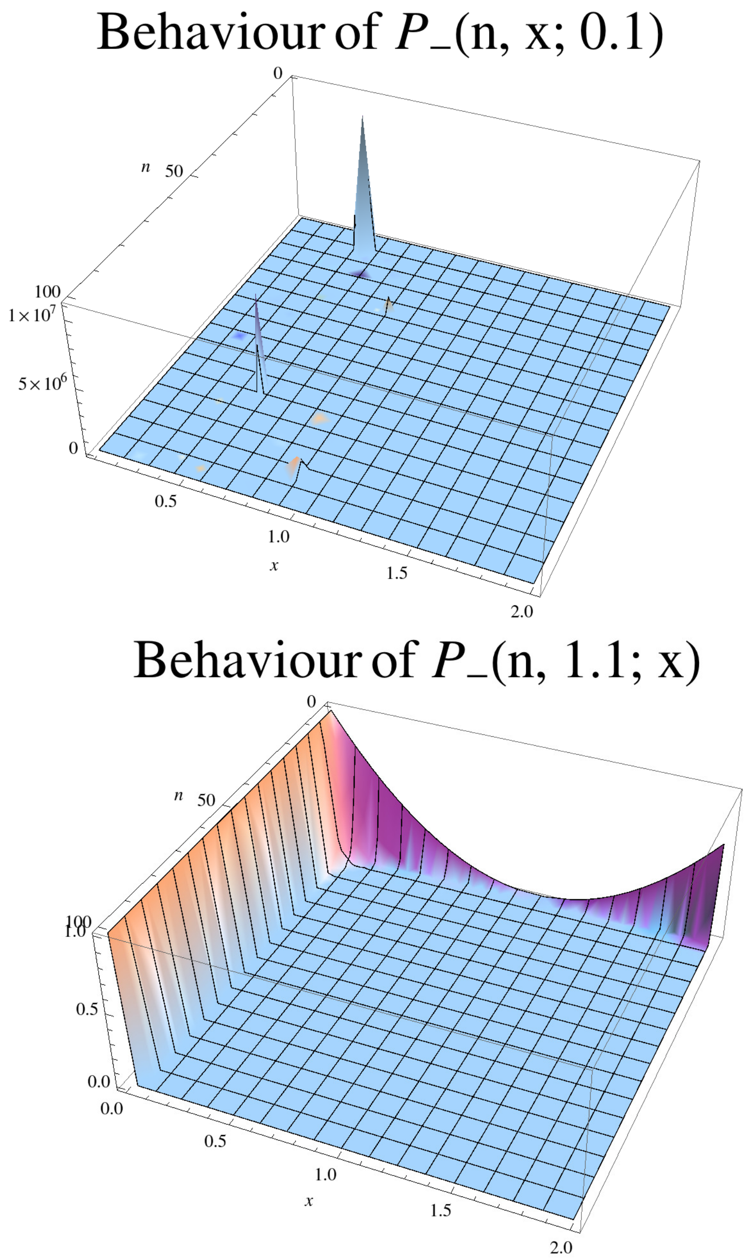


FIGURE 5.4: In this figure, we plotted $P_{-}(n, x; 0.1)$ in the upper figure and $P_{-}(n, 1.1; x)$ in the bottom one, in the region $0 < x < 2$. In both cases $P \to 0$ when $n \to \infty$ for $x > 0$.

Already in [1], the author has shown that when one applies a Shannon transform to the HCA the contributing eigenvalues, such that the continued system has interesting solutions, are limited to be those $\rho < 1$. This is a consequence of the fact that discreteness in time is equivalent to an ultraviolet cut-off on the bandwidth of the solutions.

Note that when we took the continuum limit of the HCA in preceding chapters we did it for constant eigenvalues, $\epsilon_\alpha = const$, so this limit is smooth. Therefore the oscillations of the squared norm of the components, which are a characteristic behaviour of the HCA, and are absent in the quantum case, go smoothly to 0 for $l \to 0$, $ln = t$.

5.3 An example for a two-dimensional system

Here we study a two-dimensional system ($m = 2$) with initial condition $(\psi_1(l), \psi_2(l))$ or equivalently $(\psi_+(l), \psi_-(l))$, with $\psi_-(l) = 0$. To do this, we have to rewrite the solutions (5.3).

Again, it is better to study the system using as a basis the orthonormal eigenvectors of the Hamiltonian, they will be denoted with $\{(1,0),(0,1)\}$ and they will be ordered with absolute values of the eigenvalues in descendent order $|\epsilon_1| \geq |\epsilon_2|$. Without much loss of generality, we will consider initial condition for which $\psi_+^\dagger(l)\psi_+(l) - \psi_-^\dagger(l)\psi_-(l) = 1$, recalling that this is a conserved quantity for the system. So let us write the initial conditions as:

$$\psi_+(l) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \psi_1(l) = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad \psi_2(l) = \psi_1(l), \tag{5.13}$$

with $a_i, b_i \in \mathbb{C}$ and $|a_1|^2 + |a_2|^2 = 1$. In fact, since we chose $\psi_-(l) = 0$ and, thus, $\psi_+(l) = \psi_1(l)$, we have that necessarily a_i and the b_i which are $a_i = b_i$. Now we have to consider eqs.(5.5):

$$\begin{aligned} \psi_1(t_n) &= \textstyle\sum_{\alpha=1}^2 i^n \Big(U_{n-2}(\rho_\alpha) + iU_{n-1}(\rho_\alpha)\Big) b_\alpha \ , \\ \psi_+(t_n) &= \textstyle\sum_{\alpha=1}^2 i^n \Big(U_{n-2}(\rho_\alpha) + iU_{n-1}(\rho_\alpha) - i\rho_\alpha U_{n-2}(\rho_\alpha)\Big) a_\alpha \ , \end{aligned} \tag{5.14}$$

with $\rho_\alpha = -l\epsilon_\alpha$.

We need to find the squared norm of the projections of these two vectors on the eigenvectors of the Hamiltonian. They are:

$$
\begin{aligned}
|R_1^\alpha(t_n)|^2 &= \Big|(U_{n-2}(\rho_\alpha) + iU_{n-1}(\rho_\alpha))b_\alpha\Big|^2 , \\
|R_+^\alpha(t_n)|^2 &= \Big|\Big(U_{n-2}(\rho_\alpha) + iU_{n-1}(\rho_\alpha) - i\rho_\alpha U_{n-2}(\rho_\alpha)\Big)a_\alpha\Big|^2 .
\end{aligned}
\tag{5.15}
$$

First, we will consider the two quantities above for $\rho_1 = -\rho_2 = 0.01$ and initial conditions for which we have at $t_n = l$ a state vector such hat $R_1^1(l) = R_1^2(l) = R_+^1(l) = R_+^2(l) = 0.5$, which means a $50 : 50$ mixture of both eigenstates as initial condition.

5.3.1 Numerical results

Before showing the numerical results some remark is in order. We will show the numerical evaluations of the quantities considering $n \in \mathbb{R}$ i.e. a real time variable, despite the intrinsic discreteness, $n \in \mathbb{Z}$, because the figures look nicer, in this way.

Moreover, if we apply the Shannon Theorem [22] to the discrete solutions, what we obtain are the shown continuum solutions; the informations contained in the discrete solutions are the same as in the continuum ones (which is the most important result of the Shannon Theorem).

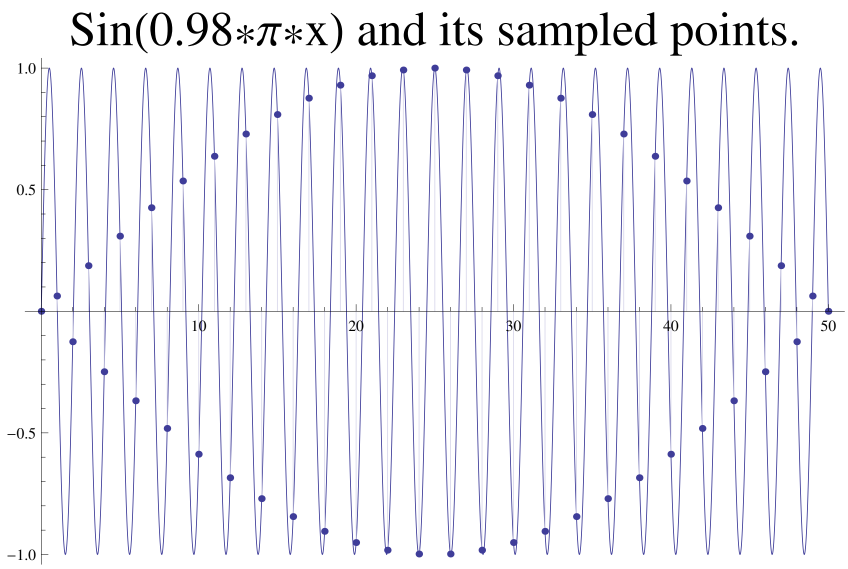


FIGURE 5.5: In this figure, we plotted $\sin(0.98\pi x)$ (the blue curve) and its discrete version sampled at the rate $r = 2\pi$ (the points). The effect of the oversampling results in the appearence of a modulation.

In some case, we will show also the discrete solutions, but only for big intervals of n.

For n real, we will find always periodical or quasi-periodical functions that have a maximum frequency less than the maximal allowed one, $\omega_{max} = \pi/l$. This means that when we consider the discrete solutions (that are sampled at the maximum rate allowed), we will see the effect of oversampling, that is the appearance of a spurious frequency.

To understand it, just consider the simple function $\sin(0.98x)$. It has period $T = 2\pi/0.98$. Now let us sample it at a rate $r = 1/\pi$ that is slightly bigger than the minimum rate, which is $0.98/\pi$. We can see the result of this sampling in fig. 5.5 in which we show both the continuum function (the line) and the discrete points.

The results of the numerical evaluation are shown in fig. 5.6. We can see that for $|\rho_1| \ll 1$ the quantities plotted in this figure ($|R_1^\alpha|^2$ and $|R_+^\alpha|^2$,respectively) are quasi constant, as for the continuum case, but with small oscillations around the average value.

Things change a little if the value of ρ_α is closer to 1. In fig. 5.7 we can see the case $\rho_1 = 0.8$ and $\rho_2 = 0.6$. The oscillations become bigger in this case. We can also note that the period varies with ρ_α.

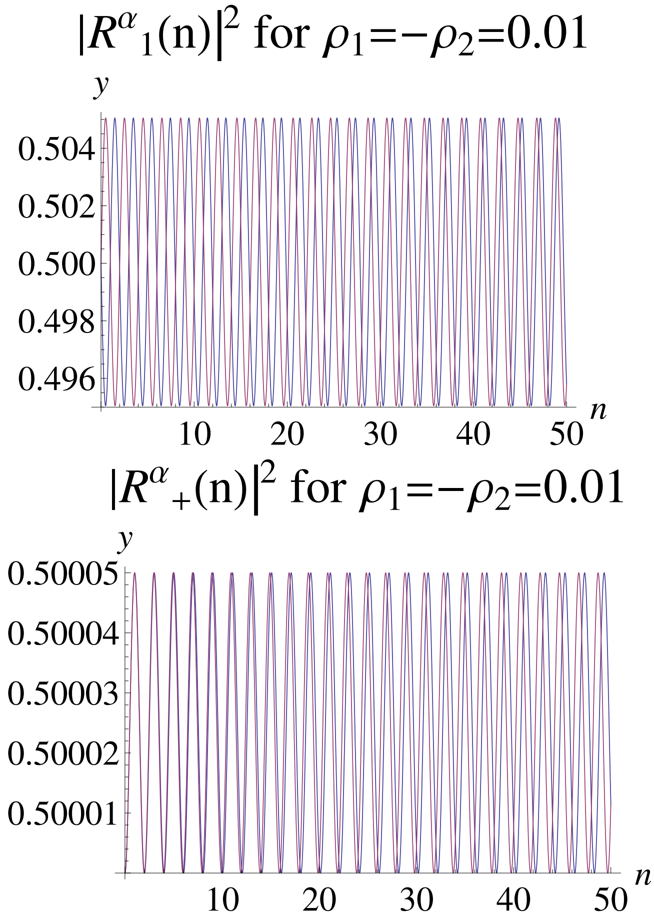


FIGURE 5.6: In this figure, we plotted $|R_1^\alpha(n)|^2$ in the upper figure and $|R_+^\alpha(n)|^2$ in the bottom one, for $\rho_1 = -\rho_2 = 0.01$. The blue curve represent $|R_1^1(n)|^2$, while the pink one represent $|R_1^2(n)|^2$. In this case, the result is a very small oscillation of the quantity around the initial value, for $|R_1^\alpha(n)|^2$, and an even smaller oscillation above the initial value for $|R_+^\alpha(n)|^2$.

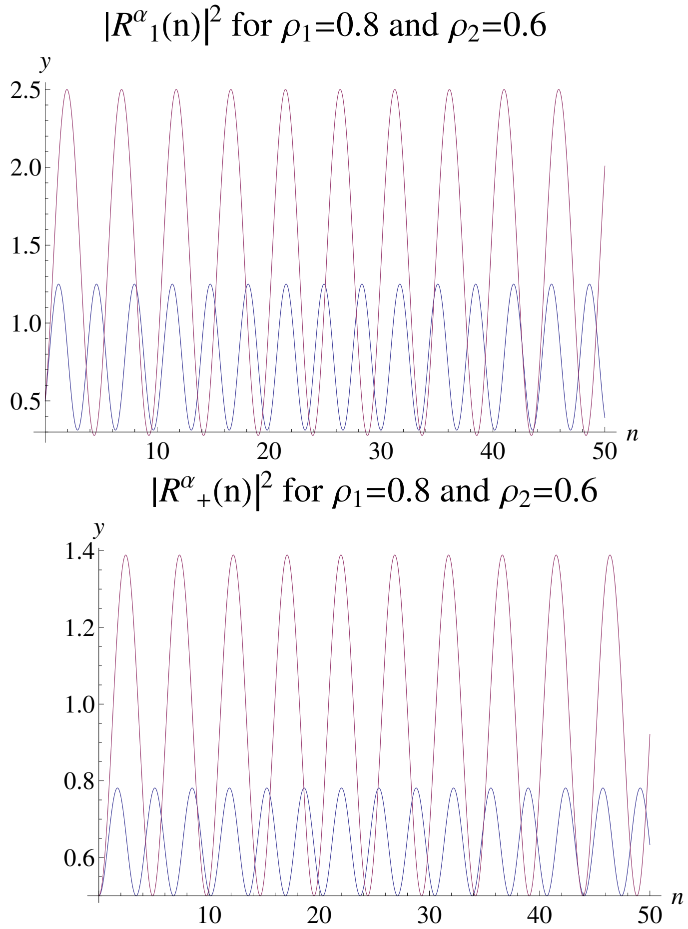


FIGURE 5.7: In this figure, we plotted $|R_1^\alpha(n)|^2$ in the upper figure and $|R_+^\alpha(n)|^2$ in the bottom one for $\rho_1 = 0.8$ (the pink curve) and $\rho_2 = 0.6$ (the blue curve). In this case, oscillations of these quantities around (or above) the initial value are quite big (of the order of the average value itself).

In fig. 5.8, we plotted for both the Hamiltonians the sum $\sum_{\alpha=1}^{2} |R_1^\alpha(t_n)|^2$, representing squared norm of the Hilbert space vectors to show that, differently from what happens in quantum mechanics, this quantity is not conserved. We can see that also this quantity oscillates periodically, with a period that depends on the difference between the two eigenvalues.

The behaviour shown in the figures above is for n real, as we said in the introduction of this chapter. The real behaviour of both R_1^α and R_+^α, when n is integer, is shown in fig. 5.9, for $\rho_1 = -\rho_2 = 0.01$. As we can see the integerness of time introduces a modulating frequency, due to the fact that the functions are taken only at integer values of n and that we are oversampling, because the frequency of the function we are looking at is not the maximum frequency allowed for the system (the frequency used to sample).

We can see that the continued-in-n function is the Shannon transform of the discrete-in-n one in the following way.

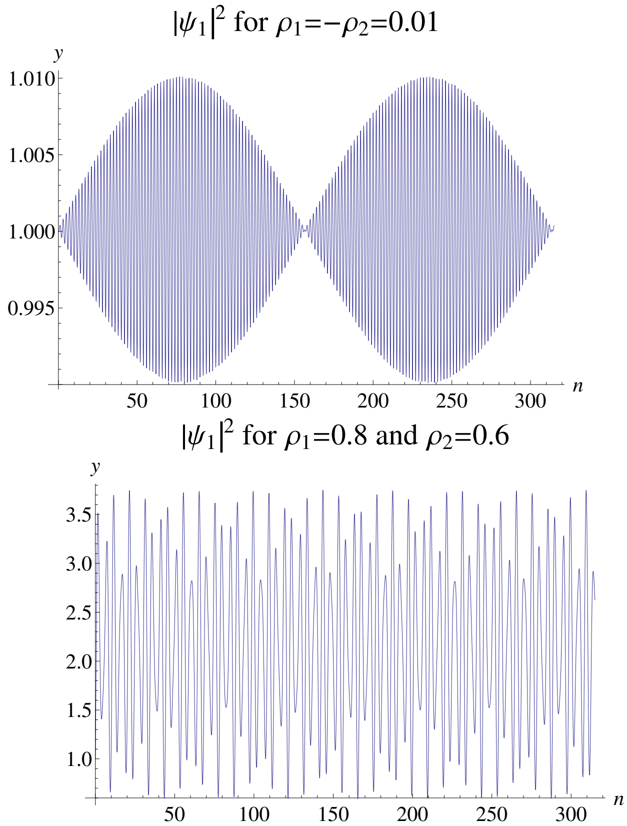


FIGURE 5.8: In this figure, we plotted $\sum_{\alpha=1}^{2} |R_1^\alpha(n)|^2$ for the case $\rho_1 = -\rho_2 = 0.01$ on the left and for $\rho_1 = 0.8$ and $\rho_2 = 0.6$ on the right.

Consider each Chebyshev polynomial as a sample at time n. Because the argument of the Chebyshev polynomials is in between -1 and 1, we can use the trigonometric definition for them. Calling $f(t)$ the continued function, according to the Shannon Theorem we have:

$$f(t) = \sum_n (iU_n + U_{n-1} - il\epsilon U_{n-1}) \frac{\sin(\pi(t/l - n))}{\pi(t/l - n)} \ . \tag{5.16}$$

In the figures above we showed the squared norm of $f(t)$, that is:

$$|f(t)|^2 = |\sum_n (iU_n + U_{n-1} - il\epsilon U_{n-1}) \frac{\sin(\pi(t/l - n))}{\pi(t/l - n)}|^2 \ . \tag{5.17}$$

Evaluating it for $t = ml$, we get:

$$|f(t_m)|^2 = |\sum_n (iU_n + U_{n-1} - il\epsilon U_{n-1}) \frac{\sin(\pi(m - n))}{\pi(m - n)}|^2 = |(iU_m + U_{m-1} - il\epsilon U_{m-1})|^2 \ , \tag{5.18}$$

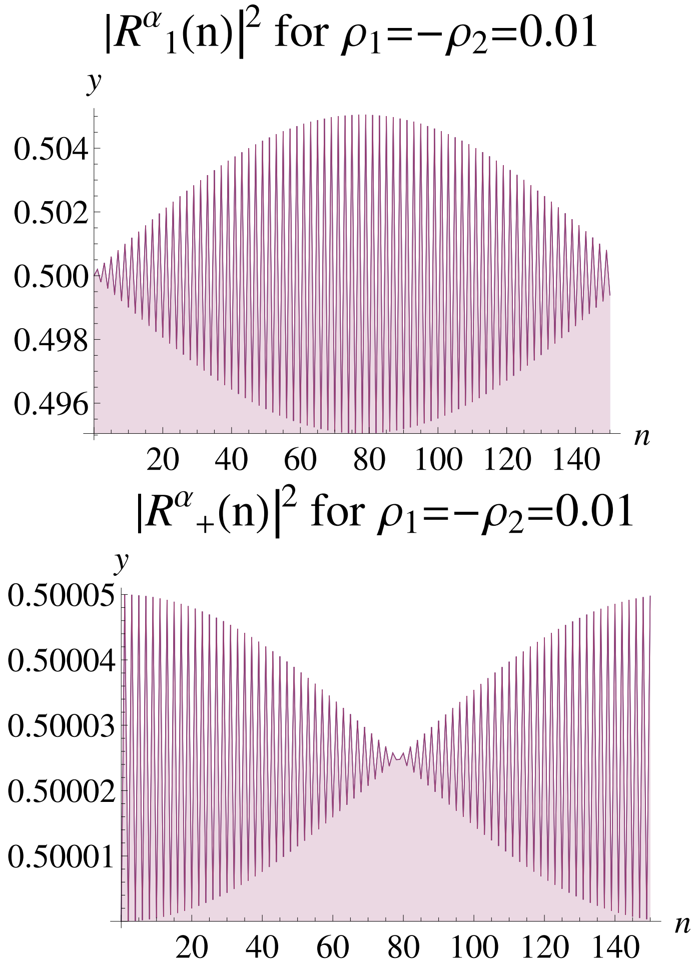


FIGURE 5.9: In this figure, we plotted $|R_1^{\alpha}(n)|^2$ in the upper figure and $|R_+^{\alpha}(n)|^2$ in the bottom one, for $\rho_1 = -\rho_2 = 0.01$. The blue curve represents $|R_{1,+}^1(n)|^2$, while the pink one represents $|R_{1,+}^2(n)|^2$. This time the function is represented only for integer values of n. A modulation appears, due to the discreteness of time, and the two curves are overlapping for $n \in \mathbb{N}$.

using the fact that $\sin(\pi(n-m))/(\pi(m-n)) = 0$ for $m \neq n$ in the second equality. These are realated to the functions of n we plotted in the discrete case (fig. 5.9).

In fig. 5.10, we plotted the constant of motion related to the identity operator, $C_{\mathbb{I}}(n, n-1) = \langle \Psi(n), \Sigma_1 \Psi_(n) \rangle = \langle \Psi'(n), \Sigma_3 \Psi'(n) \rangle$, just as a check of our numerical procedure.

To complete the study, we want to take a look also at $R_-(t_n)$. We obtain:

$$|R_-^{\alpha}(l)|^2 = \left|\rho_\alpha U_{n-2}(\rho_\alpha) b_\alpha\right|^2 . \tag{5.19}$$

Its plot for both cases, $\rho_1 = -\rho_2 = 0.01$ and $\rho_1 = 0.8$, $\rho_2 = 0.6$, is shown in fig. 5.11.

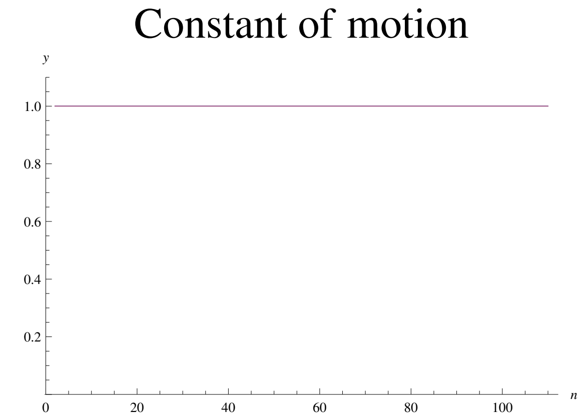


FIGURE 5.10: In this figure, we plotted the constant of motion for the observable related to the identity. In pink is the curve for $\rho_1 = -\rho_2 = 0.01$, while in blue we have the curve for $\rho_1 = 0.8$, $\rho_2 = 0.6$, because they are both constant, with the same value, we see only the pink one.

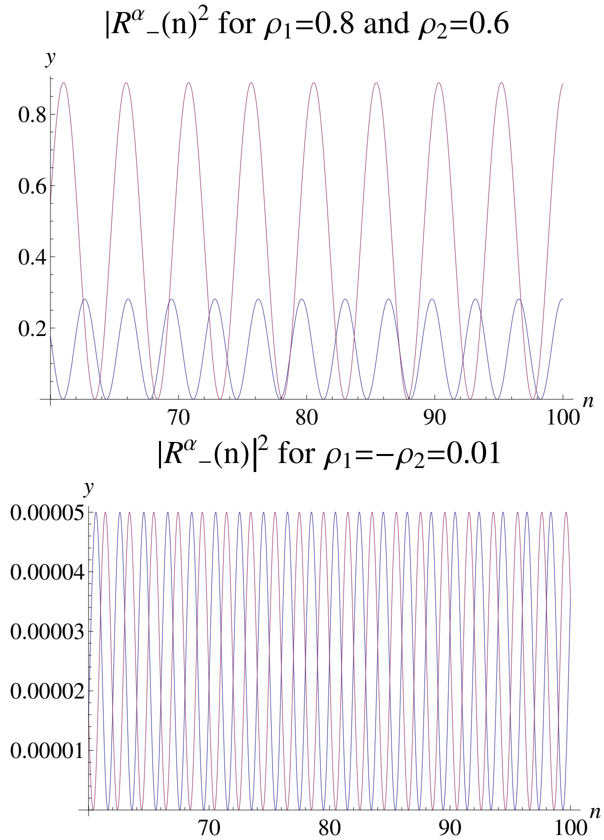


FIGURE 5.11: In this figure, we plotted $|R^{\alpha}_{-}(n)|^2$ for $\rho_1 = 0.8$, $\rho_2 = 0.6$ on the left and for $\rho_1 = -\rho_2 = 0.01$ on the right. In pink we have the curves for ρ_1 in blue those for ρ_2. Again there is an oscillating behaviour.

We can see the oscillating behaviour and the changing in the period as for both $|R^{\alpha}_{1}|^2$ and $|R^{\alpha}_{+}|^2$, but this time $|R^{\alpha}_{-}(t_n)|^2$ can take values near 0; this is why we obtained the strongly peaked behaviour in fig. 5.4.

5.3.2 The real and imaginary parts of the Hilbert space vector components

It is also interesting to look at the imaginary and real part of the components of the Hilbert space vectors, which are $\Im(R_1^\alpha(n))$ and $\Re(R_1^\alpha(n))$. They are shown in fig. 5.12 and 5.13 for the two different systems, the first with $\rho_1 = -\rho_2 = 0.1$ (which shows a more pronounced behaviour than the previously used value $\rho_1 = -\rho_2 = 0.01$) and the second with $\rho_1 = 0.8$ and $\rho_2 = 0.6$.

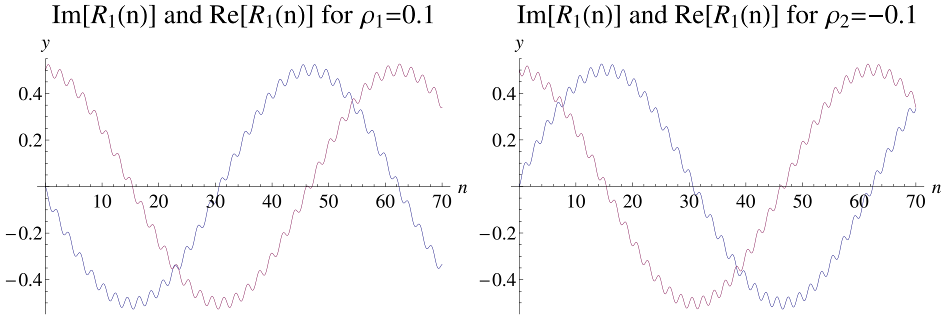


FIGURE 5.12: In this figure, we plotted $\Im(R_1^1(n))$ and $\Re(R_1^1(n))$ on the left and $\Im(R_1^2(n))$ and $\Re(R_1^2(n))$ for $\rho_1 = -\rho_2 = 0.1$ on the right. In blue we have the curves for the real parts while in pink those for the imaginary parts.

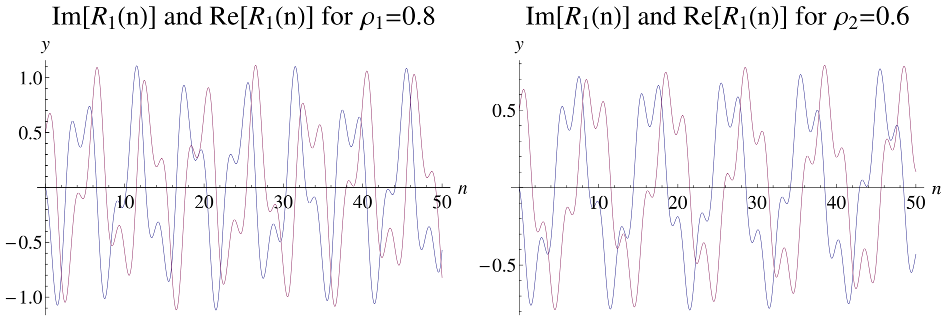


FIGURE 5.13: In this figure, we plotted $\Im(R_1^1(n))$ and $\Re(R_1^1(n))$ on the left figure and $\Im(R_1^2(n))$ and $\Re(R_1^2(n))$, for $\rho_1 = 0.8$ and $\rho_2 = 0.6$ on the right one. In blue we have the curves for the real parts, while in pink those for the imaginary parts.

In fig. 5.12 we can see that there is a short periodic oscillation superposed on a long periodic one. The long periodic oscillation is equal to the oscillation we would have in quantum mechanics, while the short periodic one is the new feauture of the HCA and is responsable for the oscillations in the squared norm of the vector components seen before. If ρ increases the short periodic oscillation increases its amplitude, while its period stays almost constant, and the long periodic one decreases its period.

To see the difference in behaviour between the continued HCA with l finite and the corresponding quantum system, we show in fig(5.14) the real and imaginary parts for one component of both systems. We can see that for small eigenvalues of the Hamiltonian the real (imaginary) part for the HCA (the rippled lines) oscillates around the value of

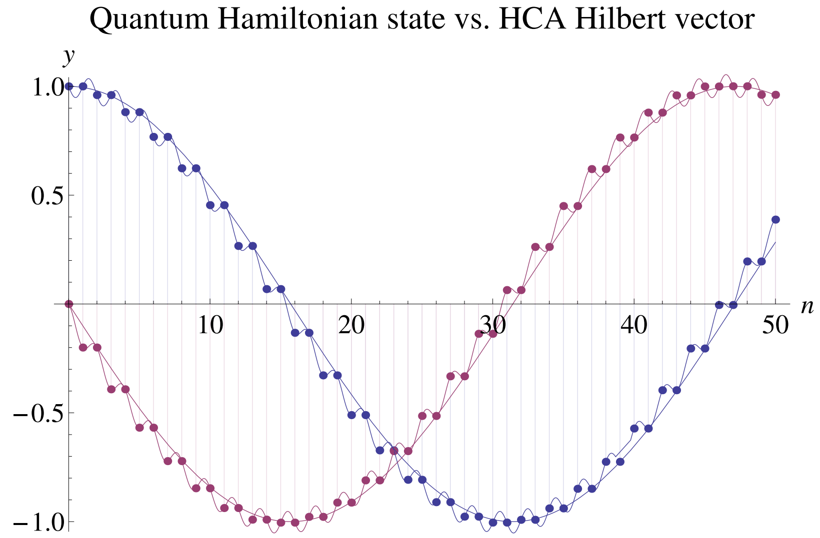


FIGURE 5.14: In this figure, we plotted the continued functions $\Re(R_1^1(n))$ (the blue rippled line) and $\Im(R_1^1(n))$ (the pink rippled line) for $\rho_1 = 0.1$, their samples at integer n (the points) and their quantum analogues $\Re(\psi^1(n))$ (the blue line) and $\Im(R_1^1(n))$ for $\epsilon_1 = 0.1$ (the pink line). We see the characteristic behaviour of the HCA Hilbert space vector as compared to the quantum state. Note that we are considering $l = 1$, so for the HCA $\rho_1 = \epsilon_1$, where ϵ_1 is the eigenvalue of the Hamiltonian of the HCA.

the real (imaginary) part of the corresponding quantum state. In the figure is also shown the discrete time real and imaginary parts of the Hilbert space vector of the HCA.

The eigenstate of the quantum system and the Hilbert space vector of the HCA are the ones corresponding to the eigenvalue $\epsilon_1 = 0.1$ (recall that we put $l = 1$).

It is worth noting that if we take the value 1 for $|\rho_1|$ as a limit, we can no more shift the zero value of the Hamiltonian, as can be done in quantum mechanics. If the HCAs have something to do with reality at a very small scale l, we can suppose that the eigenvalues of the Hamiltonians involved are very small compared to the maximum value allowed, if there is any. In this case, we can shift the zero of energy without drastically changing the behaviour.

5.4 Composite HCA systems

In the previous section, we studied the behaviour of single isolated systems, now we want to do a similar study for composite systems. We already know how a factored state $\Psi_{fac}(t_n)$ behaves under the time evolution operator, which mantains the state factored. In fact, the updating equations in the absence of an interaction, and for an initial factored state, are equivalent to those of the single systems taken separately, as we have already shown in Section (4.3.3). Thus, we find that if the initial state is factored, no spurious correlations between different parts of a composite system arise during the time evolution, if they do not interact.

Things change if we consider a general state and a general time evolution operator.

Before studying the general case, we want to consider a simpler situation: a non-factored initial state evolving with the factored time evolution operator $\hat{\mathbf{T}}_n = \hat{\mathbf{T}}'_n \otimes \hat{\mathbf{T}}''_n$, where $\hat{\mathbf{T}}_n$ is the time evolution operator of the composite system, while $\hat{\mathbf{T}}'_n$ and $\hat{\mathbf{T}}''_n$ are those of the two single systems. In particular, we consider here a state which will correspond to a composite system of two q-bits in the continuum limit. We will use $\Psi'(t_n)$, instead of using the state $\Psi(t_n)$ which we already used in Ch.(4). So, recalling what we have said in Sectio (4.23) about $\Psi'(t_n)$, we can write:

$$\Psi'(t_n) = \begin{pmatrix} \psi_{++}(t_n) \\ \psi_{+-}(t_n) \\ \psi_{-+}(t_n) \\ \psi_{--}(t_n) \end{pmatrix} . \tag{5.20}$$

Here $\psi_{++}(t_n)$ is of order $O(1)$, $\psi_{+-}(t_n)$ and $\psi_{-+}(t_n)$ are of order $O(l)$, and $\psi_{--}(t_n)$ is of order $O(l^2)$. We consider that these relations are true for the initial state. Then they will continue to hold under time evolution. The four "components" of Ψ' take their names from the case in which the state is factored. The choice of the initial conditions is compatible with the initial conditions in the case of single systems.

As in the case of single systems, we will be interested in the solution for initial conditions $\psi_{++}(l)$ and $\psi_{+-}(l) = \psi_{-+}(l) = \psi_{--}(l) = 0$, because the solution in this case is simpler and the numerical evaluation for long times does not introduce too big approximation errors. Moreover, we will consider the time evolution of the first component $\psi_{++}(t_n)$ to get an idea of what is happening, even if the information contained on the other components, being of order $O(l)$, is not neglectable, so a complete study eventually should take in account also those components.

Thus, we will start with the state:

$$\Psi'(l) = \begin{pmatrix} \psi_{++}(l) \\ 0 \\ 0 \\ 0 \end{pmatrix} , \tag{5.21}$$

To begin with, we should evaluate the time evolution operator $\hat{\mathbf{T}}_n$, but, because of what we have just said above we need only its first row first column component $(\hat{\mathbf{T}}_n)_{11}$ that is:

$$\left(\hat{\mathbf{T}}_n\right)_{11} = (\hat{\mathbf{T}}'_n)_{11} \otimes (\hat{\mathbf{T}}''_n)_{11} = [U'_{n-1} + iU'_n + il\hat{H}'U'_{n-1}] \otimes [U''_{n-1} + iU''_n + il\hat{H}''U''_{n-1}] \ , \quad (5.22)$$

where the omitted argument of U'_n is $-l\hat{H}'$ and that of U''_n is $-l\hat{H}''$, the two Hamiltonians of the single systems, respectively.

We used the same symbol as in the previous Chapter (4) for the time evolution operator, even if they are different, being written for different variables. Note also that what we called components for the states are nontheless vectors. In the simplest case they have four components (see as an example eq.(5.26)), and so the operator $(\hat{\mathbf{T}}_n)_{11}$ is represented by a 4×4 matrix, which is diagonal in the chosen basis (see eqs.(5.25)). We will study this simplest case, and we will do that in the factored orthonormal basis, the vectors of which are the tensor products between the eigenstates of the two Hamiltonians.

We will use the basis $\{\psi^1, \psi^2, \psi^3, \psi^4\}$ for the global system:

$$\begin{gathered}
\psi^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \quad \psi^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ , \\
\psi^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \quad \psi^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ .
\end{gathered} \quad (5.23)$$

The two-dimensional states are the states for the two subsystems. We will consider the case in which this two bases are made of eigenstates of the two Hamiltonians, so that we can write them as:

$$\hat{H}' = \begin{pmatrix} \epsilon'_1 & 0 \\ 0 & \epsilon'_2 \end{pmatrix} , \quad \hat{H}'' = \begin{pmatrix} \epsilon''_1 & 0 \\ 0 & \epsilon''_2 \end{pmatrix} \ . \quad (5.24)$$

The time evolution operator $(\hat{\mathbf{T}}_n)_{11}$ will be a diagonal 4×4 matrix with the diagonal components:

$$\begin{aligned}
\left[(\hat{\mathbf{T}}_n)_{11}\right]_{11} &= [U_{n-1}(-l\epsilon_1') + iU_n(-l\epsilon_1') + il\epsilon_1' U_{n-1}(-l\epsilon_1')] \\
&\quad [U_{n-1}(-l\epsilon_1'') + iU_n(-l\epsilon_1'') + il\epsilon_1'' U_{n-1}(-l\epsilon_1'')] \ , \\
\left[(\hat{\mathbf{T}}_n)_{11}\right]_{22} &= [U_{n-1}(-l\epsilon_1') + iU_n(-l\epsilon_1') + il\epsilon_1' U_{n-1}(-l\epsilon_1')] \\
&\quad [U_{n-1}(-l\epsilon_2'') + iU_n(-l\epsilon_2'') + il\epsilon_2'' U_{n-1}(-l\epsilon_2'')] \ , \\
\left[(\hat{\mathbf{T}}_n)_{11}\right]_{33} &= [U_{n-1}(-l\epsilon_2') + iU_n(-l\epsilon_2') + il\epsilon_2' U_{n-1}(-l\epsilon_2')] \\
&\quad [U_{n-1}(-l\epsilon_1'') + iU_n(-l\epsilon_1'') + il\epsilon_1'' U_{n-1}(-l\epsilon_1'')] \ , \\
\left[(\hat{\mathbf{T}}_n)_{11}\right]_{44} &= [U_{n-1}(-l\epsilon_2') + iU_n(-l\epsilon_2') + il\epsilon_2' U_{n-1}(-l\epsilon_2')] \\
&\quad [U_{n-1}(-l\epsilon_2'') + iU_n(-l\epsilon_2'') + il\epsilon_2'' U_{n-1}(-l\epsilon_2'')] \ .
\end{aligned} \tag{5.25}$$

To understand the main behaviour of the composite HCA in the case in which the initial state is not factored, we will take as initial state:

$$\psi_{++}(l) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \ . \tag{5.26}$$

We point out that this state is the analogue of the Bell state $|0\rangle|1\rangle + |1\rangle|0\rangle$, describing two entangled q-bits in the continuum limit. Considering a more general Bell state will complicate the calculation, while the study of the present case will allow us to comprehend the qualitative behaviour of the general one.

Now we can study the time evolution of the quantities:

$$\begin{aligned}
|R^2_{++}(t_n)|^2 &= \psi^{*2}_{++}(t_n)\psi^2_{++}(t_n) = \tfrac{1}{2}\left|\left[(\hat{\mathbf{T}}_{n-1})_{11}\right]_{11}\right|^2 \ , \\
|R^3_{++}(t_n)|^2 &= \psi^{*3}_{++}(t_n)\psi^3_{++}(t_n) = \tfrac{1}{2}\left|\left[(\hat{\mathbf{T}}_{n-1})_{11}\right]_{22}\right|^2 \ ,
\end{aligned} \tag{5.27}$$

which describe the time evolution of the squared absolute value of the two components of our Hilbert space vector and where $\left[(\hat{\mathbf{T}}_n)_{11}\right]_{11}$ and $\left[(\hat{\mathbf{T}}_n)_{11}\right]_{22}$ are defined in eq.(5.25).

Again we will call $\rho'_\alpha = -l\epsilon'_\alpha$ and $\rho''_\alpha = -l\epsilon''_\alpha$.

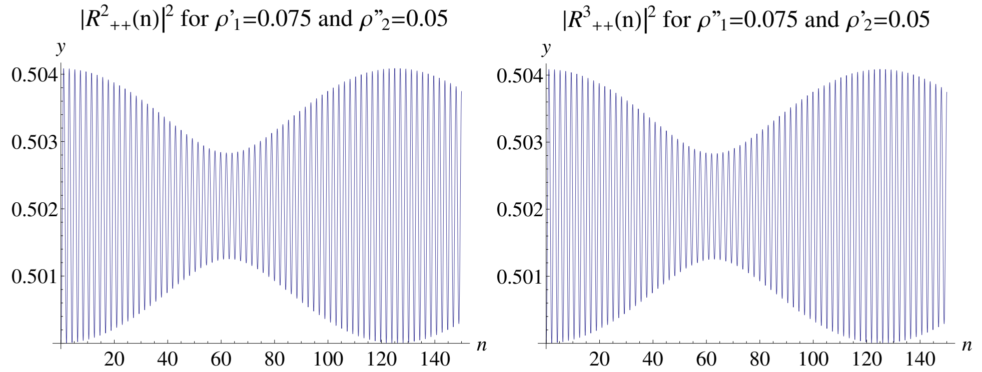


FIGURE 5.15: In this figure, we plotted $|R^2_{++}(n)|^2$ on the left and $|R^3_{++}(n)|^2$ on the right for $\rho'_1 = 0.075$, $\rho''_2 = 0.05$ and $\rho'_2 = 0.05$ and $\rho''_1 = 0.075$. We find a long period wave modulating a high frequency wave.

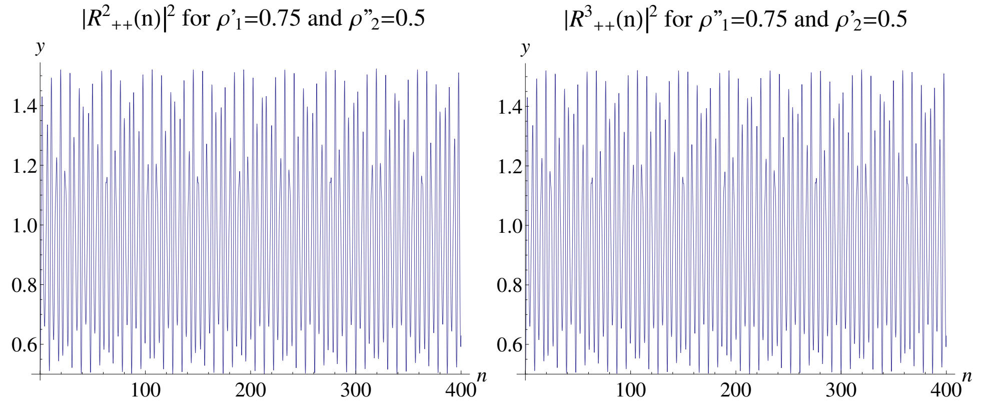


FIGURE 5.16: In this figure, we plotted $|R^2_{++}(n)|^2$ on the left and $|R^3_{++}(n)|^2$ on the right for $\rho'_1 = 0.75$, $\rho''_2 = 0.5$, $\rho'_2 = 0.5$ and $\rho''_1 = 0.75$. In this case, being the difference between the two eigenvalues not so small, we can see just a periodic behaviour.

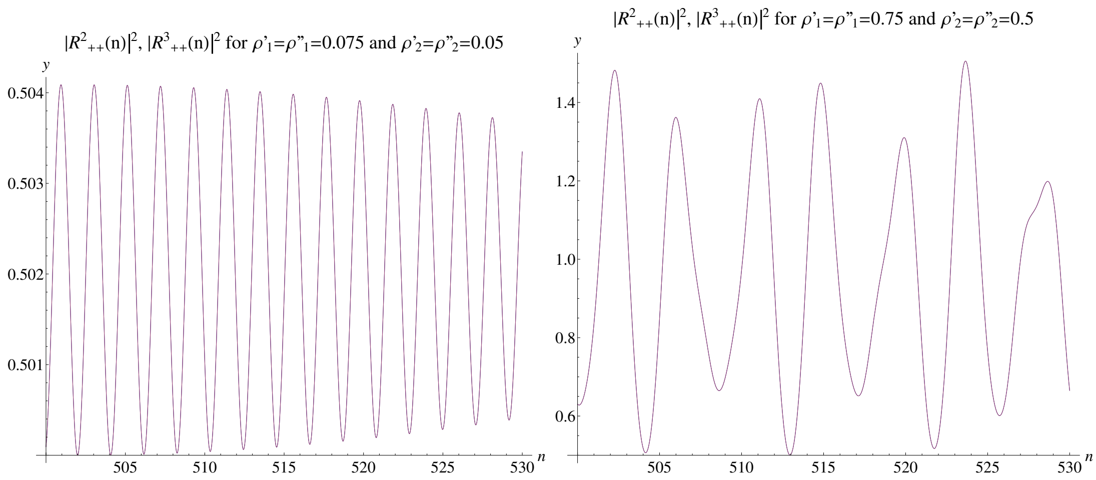


FIGURE 5.17: In this figure, we plotted $|R^2_{++}(n)|^2$ (blue curve) and $|R^3_{++}(n)|^2$ (pink curve) for $\rho'_1 = 0.075$, $\rho''_1 = 0.075$, $\rho'_2 = 0.05$ and $\rho''_2 = 0.05$, on the left and $|R^2_{++}(n)|^2$ and $|R^3_{++}(n)|^2$ for $\rho'_1 = 0.75$, $\rho''_1 = 0.75$, $\rho'_2 = 0.5$ and $\rho''_2 = 0.5$ on the right, for a small interval of time, in order to see the behaviour of the state's component in more detail.

In fig. 5.15 we show the behaviour of the quantities in (5.27) for $\rho'_1 = \rho''_1 = 0.075$ and $\rho'_2 = \rho''_2 = 0.05$. We can clearly see that the behaviour is similar to that of the single systems with small oscillations around an averaged value.

In fig. 5.16 we show the behaviour of the quantities in (5.27) for $\rho'_1 = 0.75$, $\rho''_1 = 0.75$ and $\rho''_2 = 0.5$. Also in this case the behaviour is similar to that of the single systems and, as in that case, we have bigger oscillations.

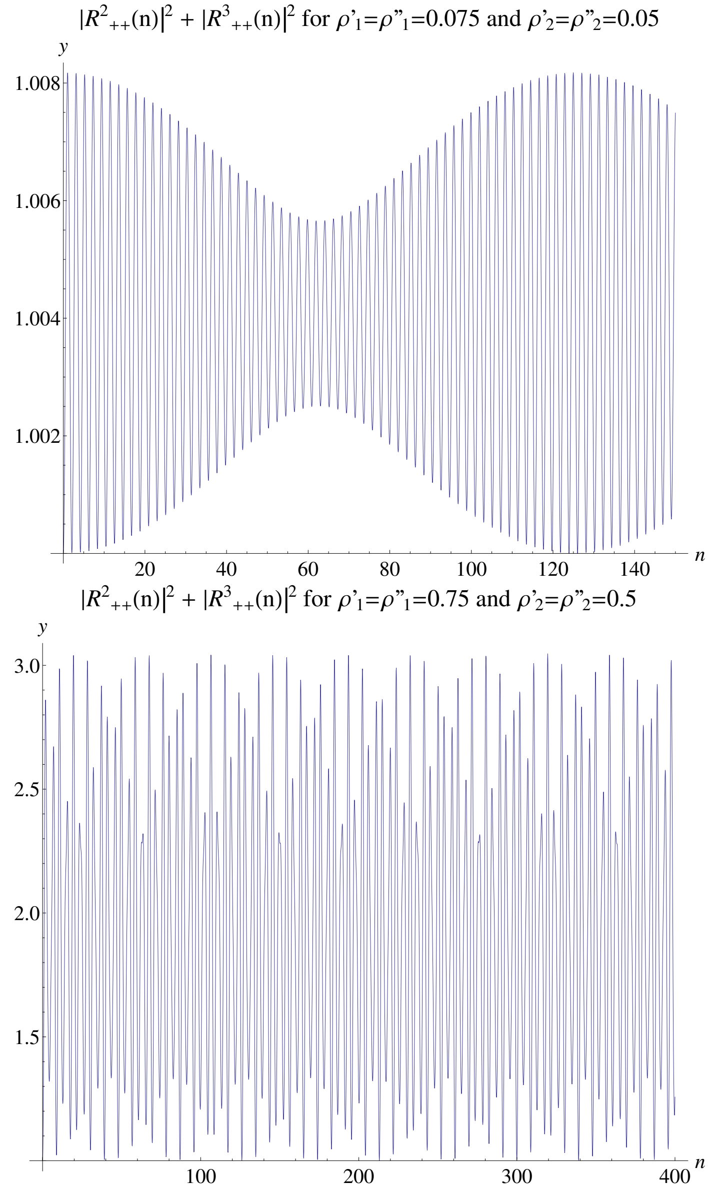


FIGURE 5.18: In this figure, we plotted $|\Psi_{++}|^2 = |R^2_{++}(n)|^2 + |R^3_{++}(n)|^2$ (the squared norm of the Hilbert space vector) for $\rho'_1 = 0.75$, $\rho''_1 = 0.75$, $\rho'_2 = 0.5$ and $\rho''_2 = 0.5$, on the left and $|R^1_{++}(n)|^2 + |R^2_{++}(n)|^2$ for $\rho'_1 = 0.075$, $\rho''_1 = 0.075$, $\rho'_2 = 0.05$ and $\rho''_2 = 0.05$ on the right. Note that its value is always bigger than 1.

In all the figures described so far, we can see that because there are two eigenvalue involved in eq.(5.27) the oscillations have two frequency components. This is more evident in fig. 5.15, because in this case, the difference is small and we can see both frequencies: one as the inner oscillations the other as the modulating wave.

In fig. 5.17 we show the behaviour of the quantities in (5.27) for both cases shown before, however, for a small interval of time, just to see with more detail what is going on.

In fig. 5.18, we plotted the squared norm of the Hilbert space vector, that is $|\psi_{++}|^2 = R^2_{++}(n) + R^3_{++}(n)$, in function of time n. We can see that, as we expected, this quantity is not a constant of motion and is always bigger than 1 for $n > 0$.

We emphasize that the initial state has not been a factored state. In fact, there is no way to write it as a tensor product of two single system states. That is true for all states with only the second and third components different from zero, and because both of these components are present for all t_n (and are the only ones) the state remains unfactored. This behaviour is in line with what happens in quantum mechanics, where a non-interacting Hamiltonian leaves a non-factored state intact, i.e. does not turn it into a factored one.

5.5 Introducing an interaction which entangles HCA states

The notations used here are those of Chapter (4) We want to find an interacting time evolution operator that can create entanglement starting from a factored state (recall that even in the interacting case we need two Hamiltonians to build it, see eq.(4.52), even if that equation is written for $\Psi(t_n)$ and not $\Psi'(t_n)$). We will consider two two-dimensional systems with states, respectively, $\Xi' = (\xi_+, \xi_-)$ and $\Phi' = (\phi_+, \phi_-)$. As before, the composite system states will be denoted by:

$$\Psi' = \begin{pmatrix} \psi_{++} \\ \psi_{+-} \\ \psi_{-+} \\ \psi_{--} \end{pmatrix} , \tag{5.28}$$

We will start with a factored state, the simplest possible, that means we will take, in the two subsystems, the states $\Xi'(l) = (\xi_+(l), 0)$ and $\Phi'(l) = (\phi_+(l), 0)$. This means that the composite system has initial state:

$$\Psi'(l) = \begin{pmatrix} \psi_{++}(l) \\ 0 \\ 0 \\ 0 \end{pmatrix} , \tag{5.29}$$

with $\psi_{++}(l) = \xi_{+}(l) \otimes \phi_{+}(l)$.

From now on, we will look just at the behaviour of $\psi_{++}(t_n)$, because even in this simple case doing the evaluation for the whole state $\Psi'(t_n)$ would be quite difficult and long. However, we can say that by chosing the initial condition in (5.29), $\psi_{+-}(t_n)$, $\psi_{-+}(t_n)$, $\psi_{--}(t_n)$ will all be of order $O(l)$ for all t_n. Naturally, this does not mean that we can really neglect them. In fact, we have to take them into account, when we want to write down the conserved quantities for the system, which we will not pursue in this section.

To find an Hamiltonian that can create entanglement it is convenient to chose a "q-bit" basis for ξ_+ and ϕ_+ , respectively $\{\xi_1, \xi_2\}$ and $\{\phi_1, \phi_2\}$, which we can represent as:

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{5.30}$$

For ψ_{++} of the composite system, we take the factored basis $\{\psi_1, \psi_2, \psi_3, \psi_4\}$ with:

$$\begin{aligned} \psi_1 = \xi_1 \otimes \phi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \psi_2 = \xi_1 \otimes \phi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} , \\ \psi_3 = \xi_2 \otimes \phi_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \psi_4 = \xi_2 \otimes \phi_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} , \end{aligned} \tag{5.31}$$

Now, before trying to guess some Hamiltonians, we need to talk about the form of entangled and factored states in this basis.

We have that all the single component states (those forming the basis) are factored and so are the states:

$$\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ a \\ 0 \\ b \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ a \\ b \end{pmatrix}, \quad \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \tag{5.32}$$

where $a, b, c, d \in \mathbb{C}$, for the last one must hold $a/b = c/d$ and $a/c = b/d$. While the entangled states are those of the kind:

$$\begin{pmatrix} 0 \\ a \\ b \\ 0 \end{pmatrix}, \quad \begin{pmatrix} a \\ 0 \\ 0 \\ b \end{pmatrix}, \quad \begin{pmatrix} a \\ b \\ c \\ 0 \end{pmatrix}, \quad \begin{pmatrix} a \\ b \\ 0 \\ c \end{pmatrix}, \quad \begin{pmatrix} a \\ 0 \\ b \\ c \end{pmatrix}, \quad \begin{pmatrix} 0 \\ a \\ b \\ c \end{pmatrix}, \tag{5.33}$$

plus the remaining four-components states which do not satisfy the conditions $a/b = c/d$ and $a/c = b/d$.

Recall that in the most general cases we need two Hamiltonians $\hat{\mathbf{H}}_1$ and $\hat{\mathbf{H}}_2$, both containing the interactions, and that we have two possible cases: (i) they commute or (ii) they do not. We will consider the commuting case for simplicity. To find a general form for $\hat{\mathbf{H}}_1$ and $\hat{\mathbf{H}}_2$ in the commuting case so that the time evolution creates entanglement we first recall that in the limit $l \to 0$, $nl = t$ the HCA corresponds to a quantum system with Hamiltonian $\hat{H} = \hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2$. In quantum machanics we can write an Hamiltonian as a sum of two parts (that do not have to correnspond to $\hat{\mathbf{H}}_1$ and $\hat{\mathbf{H}}_2$):

$$\hat{H} = \hat{K}_1 \otimes \hat{K}_2 + \hat{Q}_1 \otimes \hat{Q}_2 \, . \tag{5.34}$$

Since we need for the HCA $[\hat{\mathbf{H}}_1, \hat{\mathbf{H}}_2] = 0$ we could try to take $\hat{\mathbf{H}}_1 = \hat{\mathbf{H}}_2 = (1/2)\hat{H}$. To do the calculation we suppose, without losing generality, that in our basis both $\hat{K}_1$ and $\hat{K}_2$ are diagonal:

$$\hat{K}_1 = \begin{pmatrix} k_1' & 0 \\ 0 & k_1'' \end{pmatrix}, \quad \hat{K}_2 = \begin{pmatrix} k_2' & 0 \\ 0 & k_2'' \end{pmatrix} \, . \tag{5.35}$$

Then, to simplify the calculation further we take the following $\hat{Q}_1$ and $\hat{Q}_2$:

$$\hat{Q}_1 = \begin{pmatrix} 0 & q_1 \\ q_1^* & 0 \end{pmatrix}, \quad \hat{Q}_2 = \begin{pmatrix} 0 & q_2 \\ q_2^* & 0 \end{pmatrix} \, . \tag{5.36}$$

So that we finally obtain:

$$\hat{\mathbf{H}}_1 = \hat{\mathbf{H}}_2 = \frac{1}{2}\begin{pmatrix} k'_1k'_2 & 0 & 0 & q_1q_2 \\ 0 & k'_1k''_2 & q_1q_2^* & 0 \\ 0 & q_1^*q_2 & k''_1k'_2 & 0 \\ q_1^*q_2^* & 0 & 0 & k''_1k''_2 \end{pmatrix} . \tag{5.37}$$

To see explicitly that in this case the time evolution can create entanglement, we begin with the initial state $\psi_1 = \xi_1 \otimes \phi_1$ (see the first of eqs.(5.31)) and apply $(\hat{\mathbf{T}}_1^{\mathbf{int}})_{11}$, so we consider just one time step.

Let us write explicitly the one-time-step evolution operator $(\hat{\mathbf{T}}_1^{\mathbf{int}})_{11} = [\hat{\mathbf{I}} - il\hat{\mathbf{H}}_1][\hat{\mathbf{I}} - il\hat{\mathbf{H}}_1]$:

$$(\hat{\mathbf{T}}_1^{\mathbf{int}})_{11} =$$

$$\begin{pmatrix} A & 0 & 0 & E \\ 0 & B & F & 0 \\ 0 & F^* & C & 0 \\ E^* & 0 & 0 & D \end{pmatrix}^2 = \begin{pmatrix} A^2+|E|^2 & 0 & 0 & E(A+D) \\ 0 & B^2+|F|^2 & F(B+C) & 0 \\ 0 & F^*(B+C) & C^2+|F|^2 & 0 \\ E^*(A+D) & 0 & 0 & D^2+|E|^2 \end{pmatrix} , \tag{5.38}$$

where:

$$\begin{aligned} A &= 1 - il\frac{k'_1k'_2}{2} , \\ B &= 1 - il\frac{k'_1k''_2}{2} , \\ C &= 1 - il\frac{k''_1k'_2}{2} , \\ D &= 1 - il\frac{k''_1k''_2}{2} , \qquad (5.39) \\ E &= -il\frac{q_1q_2}{2} , \\ F &= -il\frac{q_1q_2^*}{2} \qquad (5.40) \end{aligned}$$

Now, writing down the Hilbert space vector $\psi_{++}(2l) = (\hat{\mathbf{T}}_1^{\mathbf{int}})_{11}\psi_{++}(l)$, we get:

$$\psi_{++}(2l) = (\hat{\mathbf{T}}_1^{\mathbf{int}})_{11}\psi_{++}(l) =$$

$$\begin{pmatrix} A^2+|E|^2 & 0 & 0 & E(A+D) \\ 0 & B^2+|F|^2 & F(B+C) & 0 \\ 0 & F^*(B+C) & C^2+|F|^2 & 0 \\ E^*(A+D) & 0 & 0 & D^2+|E|^2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{5.41}$$

that is:

$$\psi_{++}(2l) = \begin{pmatrix} \left(1 - il\frac{k_1'k_2'}{2}\right)^2 - l^2\frac{|q_1|^2|q_2|^2}{4} \\ 0 \\ 0 \\ -ilq_1^*q_2^* - l^2 q_1^*q_2^*\frac{k_1'k_2'+k_1''k_2''}{4} \end{pmatrix}. \tag{5.42}$$

We observe that $\psi_{++}(2l)$ is of the same kind as the second vector of the set (5.32) and so it is a non-factored (entangled) Hilbert space vector.

We have seen in these two final sections that a non-interacting Hamiltonian leaves a factored state factored and cannot turn a non-factored state into a factored one, while an interacting time evolution operator can create entanglement, as it is in quantum mechanics.

Chapter 6

Conclusions

Cellular Automata, an idealization of physical, biological, and social systems introduced by von Neumann in the late 1940s, have been extensively used to model physical systems, in particular when the interest is not in the details of the behaviour of single particles but in the global features of many-particle systems. Recently, resarchers such as G.'t Hooft [11–13] proposed a "classical interpretation" of quantum mechanics based on the idea that the fundamental underlying system is made of these kind of objects. His work has been criticized , for example in [14], and one of the main arguments is that he has not yet proposed an explicit model incorporating interactions, rather has studied quite general characteristics one could expect from such a model.

In this Thesis, we explored the features of a class of Generalized Cellular Automata, which we called Hamiltonian Cellular Automata (HCA), with the aim of understanding how to make them interact with one another.

We first have written an action for them and then derived the updating equations. They turned out to be a discretized version of Hamilton's equations (see eqs.(2.12, 2.13)). We noticed that because of the form of the action, which contains terms up to quadratic power of the variables, we could rewrite the updating equations using complex variables. Once we did this, we found equations which represent a discretized version of the Schrödinger equation, eq.(2.36):

$$\dot{\psi}_n^\alpha = -2icH_{\alpha\beta}\psi_n^\beta \ , \tag{6.1}$$

recalling that $\dot{\psi}_n = \psi_{n+1} - \psi_{n-1}$. This equation turns out to be of second order (it needs two initial conditions for each variable). At this point, we studied the conservation laws of these systems finding that they are two-point correlation functions of the kind:

$$2C_{\hat{G}}(n, n-1) = \psi_n^{*\alpha} G_{\alpha\beta} \psi_{n-1}^{\beta} + c.c. \,, \tag{6.2}$$

with $\{G_{\alpha\beta}\}$ a Hermitean matrix which commutes with the Hamiltonian $\{H_{\alpha\beta}\}$. We also introduced a minimal time step lenght l and demonstrated that in the limit $l \to 0$, $ln = t$, the complex updating equation becomes the continuum Schrödinger equation and the conserved quantities became those of quantum mechanics.

In Chapter (3), we tried to understand the space of states structure, first defining as observables all the real functions of the system variables (as in classical mechanics). Then, because of the characteristic conservation laws, we defined as observables just the quadratic functions in the variables, similarly as done by Heslot in [18], where he reformulates quantum mechanics in a Hamilton formalism. This allowed us to build a space of states structure similar to that of quantum mechanics, but with the states formed by two Hilbert space vectors instead of only one.

In order to do this, we had to study the algebraic properties of the observables, finding that they form a C*-algebra. A main difference survives between quantum systems and the HCA, namely the HCA states are not positive functionals and the C*-condition is not satisfied. Therefore, we cannot introduce, at the HCA level, an equivalent of the Born rule (a probabilistic interpretation), nor can we consider the norms of the states as density functions (deterministic interpretation). Thus, their possible role as an underlying theory for quantum mechanics needs further study, in particular its probabilistic features need to be somehow recovered.

Finally, we studied the continuum limit of the HCA in both cases (the "classical" one, for which we consider as observables all regular functions of the variables and the "quantum" one, for which the restricted observables take the form (3.54)), finding that a restriction on the initial conditions must be introduced, in order to obtain in this limit the Schrödinger equation (requiring a single initial condition) and the quantum conservation laws. In particular, the two Hilbert space vectors forming a state of the HCA must differ only by an order $O(l)$ term.

By the end of Chapter (3), we had obtained all we needed to combine two or more HCAs and, in Chapter (4), we did this for both, the "classical" and the "quantum" cases.

We have been able to compose two HCAs using the direct sum of their space of states in the "classical" case without difficulty and we showed how to build an interacting Hamiltonian.

In the "quantum" case, we used the tensor product between the spaces of states of two HCAs, first considering the non-interacting case and then introducing interactions between two HCAs. In doing this, we find that the possibilities for HCAs are richer then those for quantum mechanics and we can have different discrete systems with the same continuum limit. In particular, we saw that to build the time evolution operator for the discretized system we need two Hamiltonians, but in the continuum limit only their sum survives. Moreover, the two Hamiltonians may commute, in which case we have been able to write explicitly the time evolution operator for n time steps, or they may not commute, in which case we could not find a simple analytical form for the n-time-steps evolution operator, the discrete analogue of the familiar e^{-iHt}.

We have seen that for the HCAs, as in quantum mechanics, a non-interacting Hamiltonian does not introduce spurious unphysical correlations, while interactions can entangle non-entangled states.

Finally, in Chapter (5), we did some numerical studies in order to understand the differences between the HCA behaviour and the quantum mechanical one. We found that some restrictions on the eigenvalues of the Hamiltonian are in order, if we do not want evolving states to become unbounded in the norm. Indeed, we said that if ρ_1, i.e. minus the eigenvalue of the Hamiltonian with the biggest absolute value multiplied by l, is larger than or equal to one, then for $n \to \infty$ the main component of the state will become its projection on the eigenvector belonging to ρ_1 and the norm of the state will grow indefinitely.

We saw also that for very small eigenvalues ($\rho_1 = -l\epsilon_1 \ll 1$, with ϵ_1 the eigenvalue of $\hat{H}$), the behaviour in time of the HCA Hilbert space vectors consists in small oscillations around the corresponding quantum state. Even if these oscillations are small, they disrupt the norm conservation and this results in the impossibility of a probabilistic interpretation of the squared projections on some Hilbert space vector.

At the end of Chapter (5), we simulated also the time evolution of a composite system to show explicitly what happens to an entangled state in case of a time evolution with a non-interacting Hamiltonian. As in quantum mechanics, this kind of Hamiltonian is not able to remove the correlations between the two subsystems. Then, we used an interacting Hamiltonian to entangle a state, starting with a factored initial state.

All the simulations in the case of interacting time evolution operator are made using a time evolution operator built from two commuting Hamiltonians. It will be interesting to explore further the case of two non-commuting Hamiltonians. As we said, in this case, we have not been able to write down explicitly a simple form for the n-steps-time evolution operator, so the numerical evaluation can only be done iteratively, which is more demanding. More importantly, in this case, the observable corresponding to the energy, $\hat{\mathbf{O}}^{\mathbf{H_1}+\mathbf{H_2}}$, is not conserved.

The fact that different HCAs can describe the same quantum system in the $l \to 0$ limit is already true for single systems. In fact, because the states of the HCA are made of two Hilbert space vectors, we can always find different initial states that correspond to the same quantum state in the continuum limit. In particular, different choices of $\psi_-(l)$, do not affect the continuum limit, being $\lim_{l\to 0}\psi_-(l) = 0$. This could be explored in more detail, and one could try to see if it is possible to consider the components of ψ_- as a sort of "hidden variables" of a deterministic theory underlying quantum mechanics.

Conversely, there is also an ambiguity on how to divide the quantum Hamiltonian in order to obtain the two Hamiltonians we need for a composite HCA (for composite systems we need an Hamiltonian for each subsystem). Furthermore, there can be a qualitative difference between the behaviour of the system in the case of commuting Hamiltonians w.r.t. the case of non commuting ones. We did not explore these features of the HCA, because to do that one needs a precise and efficient way to evaluate Chebyshev polynomials of matrices.

We found in this Thesis that HCAs can be used to construct approximation schemes, provided certain restrictions on the Hamiltonian eigenvalues are fullfilled. In the first Appendix of [16] these restrictions are not respected and this is one of the reasons why the approximation fails after a few time steps).

Besides that one could try to find out if HCAs could offer an underlying deterministic theory to quantum mechanics. As we pointed out this possibility seems to be hampered by the lacking probabilistic aspects, even though it is possible that studying more accurately composite systems, mainly in the case of non-commuting Hamiltonians, can improve this.

Furthermore, the deviations of order $O(l)$ could be observable, in principle, but we first need a precise interpretation of the HCA state, since a probabilistic one seems difficult. One possible experiment one can do is to consider a state that is a mixture of two Hamiltonian eigenstates with very different eigenvalues. In this case, the squared norm of the HCA Hilbert space vector can be sensitively bigger then one, which could rule out the probabilistic interpretation.

We want also to underline that the study of composite systems with non-commuting Hamiltonians could be relevant for the measurement problem.

This completes our first study of composite systems that are formed by combining intrinsically discrete and deterministic components, namely Hamiltonian Cellular Automata. We have shown that not only the C*-algebraic structure of observables, as in quantum mechanics, can succesfully be reconstructed on such a more primitive level - than quantum mechanics based on the continuum of real or complex numbers - but also that the essential property of entanglement (related to dynamics) finds an analogue here.

Appendix A

Derivation of the equations of motion

In this appendix, we will show how to derive the equations of motion for the Hamiltonian cellular automaton (HCA).
We recall that the variables of the automaton are: x_n^α, τ_n, p_n^α, π_n, where α is an integer multi index and $n \in \mathbb{Z}$, and we employ the definitions:

$$A_n := \Delta\tau_n(H_n + H_{n-1}) + a_n \ , \tag{A.1}$$

$$H_n := \frac{1}{2}S_{\alpha\beta}(p_n^\alpha p_n^\beta + x_n^\alpha x_n^\beta) + A_{\alpha\beta}p_n^\alpha x_n^\beta \ , \tag{A.2}$$

$$a_n := c_n\pi_n \ , \tag{A.3}$$

where c_n are constants, $\hat{S} = \{S_{\alpha\beta}\}$ is a symmetric matrix, $\hat{A} = \{A_{\alpha\beta}\}$ is an antisymmetric matrix, and we introduced the notation $\Delta f_n := f_n - f_{n-1}$. The HCA action is:

$$S := \sum_n [(p_n^\alpha + p_{n-1}^\alpha)\Delta x_n^\alpha + (\pi_n + \pi_{n-1})\Delta\tau_n - A_n] \ , \tag{A.4}$$

and we will use as variation rule:

$$\delta_{f_n} g(f_n) := \frac{1}{2}[g(f_n + \delta f_n) - g(f_n - \delta f_n)] \ . \tag{A.5}$$

Varying the action with respect to the variables of the HCA we get the four equations:

$$\begin{aligned} &\delta_{x_m^\beta} S = 0 \ , \\ &\delta_{p_m^\beta} S = 0 \ , \\ &\delta_{\tau_m} S = 0 \ , \\ &\delta_{\pi_m} S = 0 \ . \end{aligned} \tag{A.6}$$

The first equation of system (A.6) gives:

$$0 = \delta_{x_m^\beta} S = \sum_n \delta_{x_m^\beta} \left[\left(p_n^\alpha + p_{n-1}^\alpha \right) \Delta x_n^\alpha \right] - \sum_n \delta_{x_m^\beta} A_n \ . \tag{A.7}$$

For the first term on the r.h.s. we have:

$$\begin{gathered} \textstyle\sum_n \delta_{x_m^\beta} \left[\left(p_n^\alpha + p_{n-1}^\alpha \right) \Delta x_n^\alpha \right] = \sum_n \left(p_n^\alpha + p_{n-1}^\alpha \right) \delta_{x_m^\beta} \Delta x_n^\alpha = \\ \textstyle\sum_n \left(p_n^\alpha + p_{n-1}^\alpha \right) \left(\delta_{mn}^{\beta\alpha} - \delta_{mn-1}^{\beta\alpha} \right) = (p_{m-1}^\beta - p_{m+1}^\beta) \delta x_m^\beta \equiv -\dot{p}_m^\beta \delta x_m^\beta \ . \end{gathered} \tag{A.8}$$

From now on we will use the notation introduced here, $\dot{O}_n = O_{n+1} - O_{n-1}$. For the second term we have:

$$\sum_n \delta_{x_m^\beta} A_n = \sum_n \Delta\tau_n \delta_{x_m^\beta} (H_n + H_{n-1}) \ , \tag{A.9}$$

where:

$$\delta_{x_m^\beta} H_n = \delta_{x_m^\beta} \left(\frac{1}{2} S_{\alpha\beta} (p_n^\alpha p_n^\beta + x_n^\alpha x_n^\beta) + A_{\alpha\beta} p_n^\alpha x_n^\beta \right) = (S_{\beta\alpha} x_n^\alpha \delta_{nm} - A_{\beta\alpha} p_n^\alpha \delta_{nm}) \delta x_m^\beta \ . \tag{A.10}$$

The summation over n in (A.9) simplifies with the two δ_{nm} to give:

$$\sum_n \delta_{x_m^\beta} A_n = (\Delta\tau_n + \Delta\tau_{n+1}) \delta_{x_m^\beta} H_n = \dot{\tau}_m \left(S_{\beta\alpha} x_m^\alpha - A_{\beta\alpha} p_m^\alpha \right) \delta x_m^\beta \ . \tag{A.11}$$

Combining eqs. (A.8) and (A.11) we get the equation of motion for p_m^β:

$$\dot{p}_m^\beta = -\dot{\tau}_m \left(S_{\beta\alpha} x_m^\alpha - A_{\beta\alpha} p_m^\alpha \right) \ . \tag{A.12}$$

The second equation of system (A.6) gives:

$$0 = \delta_{p_m^\beta} S = \sum_n \delta_{p_m^\beta} \left[\left(p_n^\alpha + p_{n-1}^\alpha \right) \Delta x_n^\alpha \right] - \sum_n \delta_{p_m^\beta} A_n \ . \tag{A.13}$$

With calculations similar to the previous case we get for the first term on the r.h.s:

$$\sum_n \delta_{p_m^\beta} \left[\left(p_n^\alpha + p_{n-1}^\alpha \right) \Delta x_n^\alpha \right] = \dot{x}_m^\beta \delta p_m^\beta \ , \tag{A.14}$$

and for the second term on the r.h.s.:

$$\textstyle\sum_n \delta_{p_m^\beta} A_n = (\Delta\tau_n + \Delta\tau_{n+1}) \delta_{p_m^\beta} H_n = \dot{\tau}_m \left(S_{\beta\alpha} p_m^\alpha + A_{\beta\alpha} x_m^\alpha \right) \delta p_m^\beta \ , \tag{A.15}$$

and combining eqs. (A.15) and (A.14) we get the equation of motion for x_m^β:

$$\dot{x}_m^\beta = \dot{\tau}_m \left(S_{\beta\alpha} p_m^\alpha + A_{\beta\alpha} x_m^\alpha \right) \ . \tag{A.16}$$

Now we are left with the variation w.r.t. the dynamical variable τ_m and its "conjugated momentum" π_m.
From the third equation of system (A.6) we get:

$$0 = \delta_{\tau_m} S = \sum_n \delta_{\tau_m} \left[\left(\pi_n + \pi_{n-1} \right) \Delta\tau_n \right] - \sum_n \delta_{\tau_m} A_n \ . \tag{A.17}$$

The first term on the r.h.s. gives simply $\dot{\pi}_m \delta\tau_m$, and the second $\dot{H}_m \delta\tau_m$ so we have:

$$\dot{\pi}_m = \dot{H}_m \ . \tag{A.18}$$

The last equation of system (A.6) gives:

$$0 = \delta_{\pi_m} S = \sum_n \delta_{\pi_m} \left[\left(\pi_n + \pi_{n-1} \right) \Delta\tau_n \right] - \sum_n \delta_{\pi_m} A_n \ , \tag{A.19}$$

that yields the updating equation for τ_m:

$$\dot{\tau}_m = c_m \ . \tag{A.20}$$

Appendix B

Time Evolution Operator (TEO) for composite HCA

In this appendix, we want to study the operator $\hat{\mathbf{T}}_n^{\mathbf{int}}$ for composite systems. First, we will show that for a composite system, if the two Hamiltonians $\hat{\mathbf{H}}_1$ and $\hat{\mathbf{H}}_2$ do commute, then $\hat{\mathbf{T}}_n^{\mathbf{int}}$ is of the form shown in eq.(4.58); then, we will prove that even in the case of two non-commuting Hamiltonians the limit $l \to 0$, $t_n = ln = t$ of $\hat{\mathbf{T}}_n^{\mathbf{int}}\Psi(l)$ is such that the components $\psi_i(t)$ of $\Psi(t)$ evolve unitarily, that is $\psi_i(t) = e^{-i(H_1+H_2)t}\psi_i$.

B.1 Finding the TEO in the commuting case

We will prove the first statement by induction. Recalling that the Chebyshev polynomial $U_{-1} = 0$, we can see that for $\hat{\mathbf{T}}_1^{\mathbf{int}}$ the statement above is true, in fact:

$$\hat{\mathbf{T}}_1^{\mathbf{int}} = \begin{pmatrix} -4c^2l^2\frac{\hat{\mathbf{H}}_1\hat{\mathbf{H}}_2+\hat{\mathbf{H}}_2\hat{\mathbf{H}}_1}{2} & -2icl\hat{\mathbf{H}}_1 & -2icl\hat{\mathbf{H}}_2 & \hat{I} \\ -2icl\hat{\mathbf{H}}_1 & 0 & \hat{I} & 0 \\ -2icl\hat{\mathbf{H}}_2 & \hat{I} & 0 & 0 \\ \hat{I} & 0 & 0 & 0 \end{pmatrix} =$$
$$- \begin{pmatrix} U_1^1U_1^2 & -iU_1^1U_0^2 & -iU_0^1U_1^2 & -U_0^1U_{n0}^2 \\ -iU_1^1U_0^2 & -U_1^1U_{-1}^2 & -U_0^1U_0^2 & iU_0^1U_{-1}^2 \\ -iU_0^1U_1^2 & -U_0^1U_0^2 & -U_{-1}^1U_1^2 & iU_{-1}^1U_0^2 \\ -U_0^1U_0^2 & iU_0^1U_{-1}^2 & iU_{-1}^1U_0^2 & U_{-1}^1U_{-1}^2 \end{pmatrix}, \tag{B.1}$$

where, as before, we omitted the argument of the Chebyshev Polynomials. Now we need to show that if eq.(4.58) it is true for n, then it is also true for $n+1$.

We have:

$$\hat{\mathbf{T}}^{\mathbf{int}}_{n+1} = \hat{\mathbf{T}}^{\mathbf{int}}_{1}\hat{\mathbf{T}}^{\mathbf{int}}_{n} \ . \tag{B.2}$$

The r.h.s. of (B.2) is:

$$\begin{pmatrix} -4c^2l^2\hat{\mathbf{H}}_1\hat{\mathbf{H}}_2 & -2icl\hat{\mathbf{H}}_1 & -2icl\hat{\mathbf{H}}_2 & \hat{I} \\ -2icl\hat{\mathbf{H}}_1 & 0 & \hat{I} & 0 \\ -2icl\hat{\mathbf{H}}_2 & \hat{I} & 0 & 0 \\ \hat{I} & 0 & 0 & 0 \end{pmatrix} \cdot$$
$$i^{2n} \begin{pmatrix} U^1_nU^2_n & -iU^1_nU^2_{n-1} & -iU^1_{n-1}U^2_n & -U^1_{n-1}U^2_{n-1} \\ -iU^1_nU^2_{n-1} & -U^1_nU^2_{n-2} & -U^1_{n-1}U^2_{n-1} & iU^1_{n-1}U^2_{n-2} \\ -iU^1_{n-1}U^2_n & -U^1_{n-1}U^2_{n-1} & -U^1_{n-2}U^2_n & iU^1_{n-2}U^2_{n-1} \\ -U^1_{n-1}U^2_{n-1} & iU^1_{n-1}U^2_{n-2} & iU^1_{n-2}U^2_{n-1} & U^1_{n-2}U^2_{n-2} \end{pmatrix} . \tag{B.3}$$

We will evaluate just $\{\hat{\mathbf{T}}^{\mathbf{int}}_{n+1}\}_{11}$ and $\{\hat{\mathbf{T}}^{\mathbf{int}}_{n+1}\}_{12}$, the other components can be evaluted in a similar way. For the first we get:

$$\begin{gathered} \{\hat{\mathbf{T}}^{\mathbf{int}}_{n+1}\}_{11} = -i^{2n}(4c^2l^2\hat{\mathbf{H}}_1\hat{\mathbf{H}}_2U^1_nU^2_n + 2cl\hat{\mathbf{H}}_1U^1_nU^2_{n-1}+ \\ 2cl\hat{\mathbf{H}}_2U^1_{n-1}U^2_n + U^1_{n-1}U^2_{n-1}) = \\ i^{2(n+1)}\left(-2cl\hat{\mathbf{H}}_1U^1_n - U^1_{n-1}\right)\left(-2cl\hat{\mathbf{H}}_2U^2_n - U^2_{n-1}\right) \ , \end{gathered} \tag{B.4}$$

where to obtain the second equality we used the fact that $[\hat{\mathbf{H}}_1, \hat{\mathbf{H}}_2] = 0$, which implies $[U^1_n, U^2_m] = 0$ for each n and m. Recalling that the argument of the Chebyshev polynomials is $-2cl\hat{\mathbf{H}}_1$ for U^1_n and $-2cl\hat{\mathbf{H}}_2$ for U^2_n and the recurrence relation for the Chebyshev polynomials of the second kind, $U_n(x) = 2xU_{n-1} - U_{n-2}$, we obtain:

$$\{\hat{\mathbf{T}}^{\mathbf{int}}_{n+1}\}_{11} = i^{2(n+1)}U^1_{n+1}U^2_{n+1} \ . \tag{B.5}$$

Similarly for $\{\hat{\mathbf{T}}^{\mathbf{int}}_{n+1}\}_{12}$ we have:

$$
\begin{aligned}
\{\hat{\mathbf{T}}^{\mathbf{int}}_{n+1}\}_{12} = i^{2n}(4ic^2l^2\hat{\mathbf{H}}_1\hat{\mathbf{H}}_2U^1_nU^2_{n-1} + 2icl\hat{\mathbf{H}}_1U^1_nU^2_{n-2}+ \\
2icl\hat{\mathbf{H}}_2U^1_{n-1}U^2_{n-1} + iU^1_{n-1}U^2_{n-2}) = \\
i^{2(n+1)}\left(-2cl\hat{\mathbf{H}}_1U^1_n - U^1_{n-1}\right)\left(-2cl\hat{\mathbf{H}}_2U^2_{n-1} - U^2_{n-2}\right) \ ,
\end{aligned}
\tag{B.6}
$$

that is:

$$
\{\hat{\mathbf{T}}^{\mathbf{int}}_{n+1}\}_{12} = i^{2(n+1)}U^1_{n+1}U^2_n \ . \tag{B.7}
$$

And similar equalities hold for each component, proving the validity of eq.(4.58).

We want to find the time evolution operator also for the case in which we use the states $\Psi'(t_n)$. We can prove that it takes the following form:

$$
\hat{\mathbf{T}}^{\mathbf{int}}_n = (-1)^{n+1}\begin{pmatrix} \hat{A}_n & \hat{B}_n & \hat{C}_n & \hat{D}_n \\ \hat{B}_n & \hat{E}_n & \hat{D}_n & \hat{F}_n \\ \hat{C}_n & \hat{D}_n & \hat{G}_n & \hat{L}_n \\ \hat{D}_n & \hat{F}_n & \hat{L}_n & \hat{M}_n \end{pmatrix} \ , \tag{B.8}
$$

where:

$$
\begin{aligned}
\hat{A}_n &= (U^1_{n-1} + iU^1_n + il\hat{H}_1U^1_{n-1})(U^2_{n-1} + iU^2_n + il\hat{H}_2U^2_{n-1}) \ , \\
\hat{B}_n &= (U^1_{n-1} + iU^1_n + il\hat{H}_1U^1_{n-1})(-il\hat{H}_2U^2_{n-1}) \ , \\
\hat{C}_n &= (-il\hat{H}_1U^1_{n-1})(U^2_{n-1} + iU^2_n + il\hat{H}_2U^2_{n-1}) \ , \\
\hat{D}_n &= (-il\hat{H}_2U^2_{n-1})(-il\hat{H}_1U^1_{n-1}) \ , \\
\hat{E}_n &= (U^1_{n-1} + iU^1_n + il\hat{H}_1U^1_{n-1})(-U^2_{n-1} + iU^2_n + il\hat{H}_2U^2_{n-1}) \ , \\
\hat{F}_n &= (-il\hat{H}_1U^1_{n-1})(-U^2_{n-1} + iU^2_n + il\hat{H}_2U^2_{n-1}) \ , \\
\hat{G}_n &= (-U^1_{n-1} + iU^1_n + il\hat{H}_1U^1_{n-1})(U^2_{n-1} + iU^2_n + il\hat{H}_2U^2_{n-1}) \ , \\
\hat{L}_n &= (-U^1_{n-1} + iU^1_n + il\hat{H}_1U^1_{n-1})(-il\hat{H}_2U^2_{n-1}) \ , \\
\hat{M}_n &= (-U^1_{n-1} + iU^1_n + il\hat{H}_1U^1_{n-1})(-U^2_{n-1} + iU^2_n + il\hat{H}_2U^2_{n-1}) \ .
\end{aligned}
\tag{B.9}
$$

To prove this, we first note that $\hat{\mathbf{T}}_1^{\mathbf{int}}$ is of this kind, then we suppose that for $\hat{\mathbf{T}}_n^{\mathbf{int}}$ holds (B.13) and we show that this implies that (B.13) holds also for $\hat{\mathbf{T}}_{n+1}^{\mathbf{int}}$. As before, we will evaluate explicitly just the term $\hat{A}_{n+1}$. The other terms can be evaluated in a similar manner. Because $\hat{\mathbf{T}}_{n+1}^{\mathbf{int}} = \hat{\mathbf{T}}_1^{\mathbf{int}}\hat{\mathbf{T}}_n^{\mathbf{int}}$, to evaluate $\hat{A}_{n+1}$, we just need $\hat{A}_1$, $\hat{B}_1$, $\hat{C}_1$, $\hat{D}_1$ and $\hat{A}_n$, $\hat{B}_n$, $\hat{C}_n$, $\hat{D}_n$, and we have:

$$\begin{gathered}
\hat{A}_{n+1} = -\hat{A}_1\hat{A}_n - \hat{B}_1\hat{B}_n - \hat{C}_1\hat{C}_n - \hat{D}_1\hat{D}_n = \\
(U_n^1 + iU_{n+1}^1 + il\hat{H}_1U_n^1 - il^2(\hat{H}_1)^2U_{n-1}^1)(U_n^2 + iU_{n+1}^2 + il\hat{H}_2U_n^2 - il^2(\hat{H}_2)^2U_{n-1}^2) + \\
(U_n^1 + iU_{n+1}^1 + il\hat{H}_1U_n^1 - il^2(\hat{H}_1)^2U_{n-1}^1)(il^2(\hat{H}_2)^2U_{n-1}^2) + \\
(il^2(\hat{H}_1)^2U_{n-1}^1)(U_n^2 + iU_{n+1}^2 + il\hat{H}_2U_n^2 - il^2(\hat{H}_2)^2U_{n-1}^2) + \\
-l^4(\hat{H}_1\hat{H}_2)^2U_{n-1}^1U_{n-1}^2 \ ,
\end{gathered} \tag{B.10}$$

and, after some calculation, we get:

$$\hat{A}_{n+1} = (U_n^1 + iU_{n+1}^1 + il\hat{H}_1U_n^1)(U_n^2 + iU_{n+1}^2 + il\hat{H}_2U_n^2) \ , \tag{B.11}$$

This is of the same form of $\hat{A}_n$ in (B.13). The same is true for the other matrices in (B.13) and (B.14), so the proof is complete.

B.2 The limit of the TEO in the non-commuting case

We have been able to write the time evolution operator $\hat{\mathbf{T}}_n^{\mathbf{int}}$, in the case $[\hat{\mathbf{H}}_1, \hat{\mathbf{H}}_2] = 0$, in a simplified form using the Chebyshev polynomials and we showed what happens in the limit $l \to 0$, $t_n = t$. It is not possible to do that for the more general case of non-commuting Hamiltonians, so we need another method to find that limit.

It is useful to decompose $\hat{\mathbf{T}}_n^{\mathbf{int}}$ as a sum of operators of different order in l and to put toghether the operators of order $O(1)$ and $O(l)$ in a single operator that we will call $\hat{\mathbf{T}}_n^{\mathbf{int(0+1)}}$, so we will write:

$$\hat{\mathbf{T}}_n^{\mathbf{int}} = \sum_{j=0}^{2n} l^j \hat{\mathbf{T}}_n^{\mathbf{int(j)}} = \hat{\mathbf{T}}_n^{\mathbf{int(0+1)}} + O(l^2) \ . \tag{B.12}$$

Then we can use the fact that:

$$\lim_{l\to 0}\hat{\mathbf{T}}_n^{\mathbf{int}}\Psi(l) = \lim_{l\to 0}\hat{\mathbf{T}}_n^{\mathbf{int(0+1)}}\Psi(l)\ . \tag{B.13}$$

For $\hat{\mathbf{T}}_1^{\mathbf{int}}$ we have $\hat{\mathbf{T}}_1^{\mathbf{int}} = \hat{\mathbf{T}}_1^{\mathbf{int(0+1)}} + O(l^2)$ and recalling that $\hat{\mathbf{T}}_n^{\mathbf{int}} = (\hat{\mathbf{T}}_1^{\mathbf{int}})^n$ we can evaluate $\hat{\mathbf{T}}_n^{\mathbf{int(0+1)}}$ for each n.

We will first find the $0^{th}+1^{st}$ order time evolution operator for $n=2k$, $k\in\mathbb{N}$, then we will find it for the odd case using the fact that $\hat{\mathbf{T}}_{2k+1}^{\mathbf{int(0+1)}} = \hat{\mathbf{T}}_1^{\mathbf{int(0+1)}}\hat{\mathbf{T}}_{2k}^{\mathbf{int(0+1)}}$.

We can prove by induction that:

$$\hat{\mathbf{T}}_{2k}^{\mathbf{int(0+1)}} = \begin{pmatrix} \hat{I} & -i2klc\hat{\mathbf{H}}_2 & -i2klc\hat{\mathbf{H}}_1 & 0 \\ -i2klc\hat{\mathbf{H}}_2 & \hat{I} & 0 & -i2klc\hat{\mathbf{H}}_1 \\ -i2klc\hat{\mathbf{H}}_1 & 0 & \hat{I} & -i2klc\hat{\mathbf{H}}_2 \\ 0 & -i2klc\hat{\mathbf{H}}_1 & -i2klc\hat{\mathbf{H}}_2 & \hat{I} \end{pmatrix}, \tag{B.14}$$

in fact, for $k=1$ we have:

$$\hat{\mathbf{T}}_{2}^{\mathbf{int(0+1)}} = \begin{pmatrix} \hat{I} & -i2lc\hat{\mathbf{H}}_2 & -i2lc\hat{\mathbf{H}}_1 & 0 \\ -i2lc\hat{\mathbf{H}}_2 & \hat{I} & 0 & -i2lc\hat{\mathbf{H}}_1 \\ -i2lc\hat{\mathbf{H}}_1 & 0 & \hat{I} & -i2lc\hat{\mathbf{H}}_2 \\ 0 & -i2lc\hat{\mathbf{H}}_1 & -i2lc\hat{\mathbf{H}}_2 & \hat{I} \end{pmatrix}, \tag{B.15}$$

then we have to show that if eq.(B.14) holds for k, then it holds also for $k+1$. So we have to consider:

$$\hat{\mathbf{T}}_{2(k+1)}^{\mathbf{int(0+1)}} = \left(\hat{\mathbf{T}}_{2}^{\mathbf{int(0+1)}}\hat{\mathbf{T}}_{2k}^{\mathbf{int(0+1)}}\right)^{(0+1)}\ , \tag{B.16}$$

where the superscript $(0+1)$ on the r.h.s. means that we must keep just the 0^{th} and 1^{st} order of the term in brackets. We have:

$$\hat{\mathbf{T}}_2^{\mathbf{int(0+1)}}\hat{\mathbf{T}}_{2k}^{\mathbf{int(0+1)}} =$$

$$\begin{pmatrix} \hat{I} & -i2lc\hat{\mathbf{H}}_2 & -i2lc\hat{\mathbf{H}}_1 & 0 \\ -i2lc\hat{\mathbf{H}}_2 & \hat{I} & 0 & -i2lc\hat{\mathbf{H}}_1 \\ -i2lc\hat{\mathbf{H}}_1 & 0 & \hat{I} & -i2lc\hat{\mathbf{H}}_2 \\ 0 & -i2lc\hat{\mathbf{H}}_1 & -i2lc\hat{\mathbf{H}}_2 & \hat{I} \end{pmatrix}.$$

$$\begin{pmatrix} \hat{I} & -i2klc\hat{\mathbf{H}}_2 & -i2klc\hat{\mathbf{H}}_1 & 0 \\ -i2klc\hat{\mathbf{H}}_2 & \hat{I} & 0 & -i2klc\hat{\mathbf{H}}_1 \\ -i2klc\hat{\mathbf{H}}_1 & 0 & \hat{I} & -i2klc\hat{\mathbf{H}}_2 \\ 0 & -i2klc\hat{\mathbf{H}}_1 & -i2klc\hat{\mathbf{H}}_2 & \hat{I} \end{pmatrix} =$$

$$\begin{pmatrix} \hat{I}+O(l^2) & -i2cl\hat{\mathbf{H}}_2 - i2kcl\hat{\mathbf{H}}_2 & -i2cl\hat{\mathbf{H}}_1 - i2kcl\hat{\mathbf{H}}_1 & O(l^2) \\ -i2cl\hat{\mathbf{H}}_2 - i2kcl\hat{\mathbf{H}}_2 & \hat{I}+O(l^2) & O(l^2) & -i2cl\hat{\mathbf{H}}_1 - i2kcl\hat{\mathbf{H}}_1 \\ -i2cl\hat{\mathbf{H}}_1 - i2kcl\hat{\mathbf{H}}_1 & O(l^2) & \hat{I}+O(l^2) & -i2cl\hat{\mathbf{H}}_2 - i2kcl\hat{\mathbf{H}}_2 \\ O(l^2) & -i2cl\hat{\mathbf{H}}_1 - i2kcl\hat{\mathbf{H}}_1 & -i2cl\hat{\mathbf{H}}_2 - i2kcl\hat{\mathbf{H}}_2 & \hat{I}+O(l^2) \end{pmatrix}, \tag{B.17}$$

which means that for $\hat{\mathbf{T}}_{2(k+1)}^{\mathbf{int(0+1)}}$ we can write:

$$\hat{\mathbf{T}}_{2(k+1)}^{\mathbf{int(0+1)}} =$$

$$\begin{pmatrix} \hat{I} & -i2(k+1)lc\hat{\mathbf{H}}_2 & -i2(k+1)lc\hat{\mathbf{H}}_1 & 0 \\ -i2(k+1)lc\hat{\mathbf{H}}_2 & \hat{I} & 0 & -i2(k+1)lc\hat{\mathbf{H}}_1 \\ -i2(k+1)lc\hat{\mathbf{H}}_1 & 0 & \hat{I} & -i2(k+1)lc\hat{\mathbf{H}}_2 \\ 0 & -i2(k+1)lc\hat{\mathbf{H}}_1 & -i2(k+1)lc\hat{\mathbf{H}}_2 & \hat{I} \end{pmatrix}, \tag{B.18}$$

as needed.

Now it is easy to write $\hat{\mathbf{T}}_{2k+1}^{\mathbf{int(0+1)}}$ using the fact that:

$$\hat{\mathbf{T}}_{2k+1}^{\mathbf{int(0+1)}} = \left(\hat{\mathbf{T}}_1^{\mathbf{int(0+1)}}\hat{\mathbf{T}}_{2k}^{\mathbf{int(0+1)}}\right)^{(0+1)}, \tag{B.19}$$

where again the superscript $(0+1)$ means that we shall keep only the 0^{th} and 1^{st} orders in l. We have:

$$\hat{\mathbf{T}}_1^{\mathbf{int(0+1)}}\hat{\mathbf{T}}_{2k}^{\mathbf{int(0+1)}} =$$

$$\begin{pmatrix} 0 & -2icl\hat{\mathbf{H}}_1 & -2icl\hat{\mathbf{H}}_2 & \hat{I} \\ -2icl\hat{\mathbf{H}}_1 & 0 & \hat{I} & 0 \\ -2icl\hat{\mathbf{H}}_2 & \hat{I} & 0 & 0 \\ \hat{I} & 0 & 0 & 0 \end{pmatrix}.$$

$$\begin{pmatrix} \hat{I} & -i2klc\hat{\mathbf{H}}_2 & -i2klc\hat{\mathbf{H}}_1 & 0 \\ -i2klc\hat{\mathbf{H}}_2 & \hat{I} & 0 & -i2klc\hat{\mathbf{H}}_1 \\ -i2klc\hat{\mathbf{H}}_1 & 0 & \hat{I} & -i2klc\hat{\mathbf{H}}_2 \\ 0 & -i2klc\hat{\mathbf{H}}_1 & -i2klc\hat{\mathbf{H}}_2 & \hat{I} \end{pmatrix} =$$

$$\begin{pmatrix} O(l^2) & -i2lc\hat{\mathbf{H}}_1 - i2klc\hat{\mathbf{H}}_2 & -i2lc\hat{\mathbf{H}}_2 - i2klc\hat{\mathbf{H}}_1 & \hat{I} + O(l^2) \\ -i2cl\hat{\mathbf{H}}_1 - i2kcl\hat{\mathbf{H}}_2 & O(l^2) & \hat{I} + O(l^2) & -i2lc\hat{\mathbf{H}}_2 - i2kcl\hat{\mathbf{H}}_1 \\ -i2lc\hat{\mathbf{H}}_2 - i2kcl\hat{\mathbf{H}}_1 & \hat{I} + O(l^2) & O(l^2) & -i2lc\hat{\mathbf{H}}_1 - i2klc\hat{\mathbf{H}}_2 \\ \hat{I} + O(l^2) & -i2lc\hat{\mathbf{H}}_2 - i2klc\hat{\mathbf{H}}_1 & -i2lc\hat{\mathbf{H}}_1 - i2klc\hat{\mathbf{H}}_2 & O(l^2) \end{pmatrix}, \tag{B.20}$$

which means that:

$$\hat{\mathbf{T}}_{2k+1}^{\mathbf{int(0+1)}} =$$

$$\begin{pmatrix} 0 & -i2lc(\hat{\mathbf{H}}_1 + k\hat{\mathbf{H}}_2) & -i2lc(\hat{\mathbf{H}}_2 + k\hat{\mathbf{H}}_1) & \hat{I} \\ -i2lc(\hat{\mathbf{H}}_1 + k\hat{\mathbf{H}}_2) & 0 & \hat{I} & -i2lc(\hat{\mathbf{H}}_2 + k\hat{\mathbf{H}}_1) \\ -i2lc(\hat{\mathbf{H}}_2 + k\hat{\mathbf{H}}_1) & \hat{I} & 0 & -i2lc(\hat{\mathbf{H}}_1 + k\hat{\mathbf{H}}_2) \\ \hat{I} & -i2lc(\hat{\mathbf{H}}_2 + k\hat{\mathbf{H}}_1) & -i2lc(\hat{\mathbf{H}}_1 + k\hat{\mathbf{H}}_2) & 0 \end{pmatrix}. \tag{B.21}$$

So now we have for both n odd and n even the time evolution operator until $O(l)$ and we can evaluate the limit of $\hat{\mathbf{T}}_n^{\mathbf{int}}\Psi(l)$ for $l \to 0$, $t_n = nl = t$.

In the limit $l \to 0$, we have for the initial condition $\Psi(l)$:

$$\lim_{l\to 0}\Psi(l) = \lim_{l\to 0}\begin{pmatrix} \psi_1(0) + l\chi_1 \\ \psi_1(0) + l\chi_2 \\ \psi_1(0) + l\chi_3 \\ \psi_1(0) + l\chi_4 \end{pmatrix} = \lim_{l\to 0}\Psi(0) + l\chi\ , \tag{B.22}$$

where we kept the first order in l for consistency, because we have to evaluate the limit of $\hat{\mathbf{T}}_n^{\mathbf{int}}\Psi(l)$, and in order to do that we shall take into account the 0^{th} and 1^{st} order of

$\hat{\mathbf{T}}_n^{\mathbf{int}}$. We can easily see that we can neglect $l\chi$. In fact, the lower order of $\hat{\mathbf{T}}_n^{\mathbf{int}} l\chi$ is of order $O(l)$, but it does not depend on n, so in the limit goes to zero (remember that the limit is done with $ln = t$).

So we can evaluate just:

$$\lim_{l\to 0} \hat{\mathbf{T}}_n^{\mathbf{int}} \Psi(l) = \lim_{l\to 0} \hat{\mathbf{T}}_n^{\mathbf{int(0+1)}} \Psi(0) \ . \tag{B.23}$$

To do the calculation, we need to consider the odd and even cases separately. We start with $n = 2k$. We get for the components of $\lim l \to 0 \Psi t_n + l = \Psi(t)$:

$$\begin{aligned}
\psi_1(t) &= \lim_{l\to 0} \left(\hat{I} - i2klc(\hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2)\right) \psi_1(0) = e^{-ic(\hat{\mathbf{H}}_1+\hat{\mathbf{H}}_2)t}\psi_1 \\
\psi_2(t) &= \lim_{l\to 0} \left(\hat{I} - i2klc(\hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2)\right) \psi_1(0) = e^{-ic(\hat{\mathbf{H}}_1+\hat{\mathbf{H}}_2)t}\psi_1 \\
\psi_3(t) &= \lim_{l\to 0} \left(\hat{I} - i2klc(\hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2)\right) \psi_1(0) = e^{-ic(\hat{\mathbf{H}}_1+\hat{\mathbf{H}}_2)t}\psi_1 \\
\psi_4(t) &= \lim_{l\to 0} \left(\hat{I} - i2klc(\hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2)\right) \psi_1(0) = e^{-ic(\hat{\mathbf{H}}_1+\hat{\mathbf{H}}_2)t}\psi_1 \ .
\end{aligned} \tag{B.24}$$

So we get four identical components as in the commuting case, all of them evolving unitarily in time. The case of n odd gives the same result, in fact, the only difference with eqs.(B.24) is that $2kl$ is replaced by $(2k + 2)l$ and we have $\lim_{l\to 0}(2k + 2)l = \lim_{l\to 0} 2kl = t$. Obviously, because of the redundancy of the information given by $\Psi(t)$, we can do the same consideration we have done here for the commuting case, as well.

Bibliography

[1] H.-T. Elze. Action principle for cellular automata and the linearity of quantum mechanics. *Phys. Rev. A*, 89:012111, Jan 2014.

[2] B. Chopard and M. Droz. *Cellular automata modeling of physical systems*, volume 24. Cambridge University Press, 1998.

[3] T. Toffoli and N. Margolus. *Cellular Automata Machines: A New Environment for Modeling*. The MIT Press, 1987.

[4] P. W. Braun H. Balzter and W. Köhler. Cellular automata models for vegetation dynamics. *Ecological Modelling*, 107(2–3):113 – 125, 1998.

[5] O. Roblin N. Boccara and M. Roger. Automata network predator-prey model with pursuit and evasion. *Phys. Rev. E*, 50:4531–4541, Dec 1994.

[6] V. Mardiris I. Karafyllidis N. Glykos Ch. Mizas, G.Ch. Sirakoulis and R. Sandaltzopoulos. Reconstruction of DNA sequences using genetic algorithms and cellular automata: Towards mutation prediction? *Biosystems*, 92(1):61 – 68, 2008.

[7] A. A. Patel, E. T. Gawlinski, S. K. Lemieux, and R. A. Gatenby. A cellular automaton model of early tumor growth and invasion: The effects of native tissue vascularity and increased anaerobic tumor metabolism. *Journal of Theoretical Biology*, 213(3):315 – 331, 2001.

[8] D. M. Messick and W. B. G. Liebrand. Individual heuristics and the dynamics of cooperation in large groups. *Psychological Review*, 102(1):131, 1995.

[9] J. G. Polhill N. M. Gotts and A. N. R. Law. Agent-based simulation in the study of social dilemmas. *Artif. Intell. Rev.*, 19(1):3–92, March 2003.

[10] Y. Oono and M. Kohmoto. Discrete model of chemical turbulence. *Phys. Rev. Lett.*, 55:2927–2931, Dec 1985.

[11] G. 't Hooft. The Fate of Quantum. *preprint arXiv:1308.1007*, 2013.

[12] G. 't Hooft. The cellular automaton interpretation of quantum mechanics. a view on the quantum nature of our universe, compulsory or impossible? *arXiv preprint arXiv:1405.1548*, 2014.

[13] G. 't Hooft. How a wave function can collapse without violating Schrödinger equation, and how to understand Born's rule. *preprint arXiv:1112.1811v3*, 2012.

[14] M. Luboš. http://motls.blogspot.it/2014/05/there-are-no-t-hoofts-ontological-bases.html. 2014-5-19.

[15] T. D. Lee. Can time be a discrete dynamical variable? *Physics Letters B*, 122(3–4):217 – 220, 1983.

[16] D. Gigli. Application of Shannon's Sampling Theorem in Quantum Mechanics. Master's thesis, Università di Pisa, Italy, 2014.

[17] F. Casagrande and E. Montaldi. Some Remarks on Finite-Difference Equation of Physical Interest. *Il Nuovo Cimento*, 40 A(4):369 – 381, 1977.

[18] A. Heslot. Quantum mechanics as a classical theory. *Phys. Rev. D*, 31:1341–1348, Mar 1985.

[19] F. Strocchi. *An introduction to the mathematical structure of quantum mechanics*. World Scientific, 2008.

[20] J. Bognár. *Indefinite inner product spaces*, volume 25. Springer Berlin, 1974.

[21] O. Bratteli and D.W. Robinson. *Operator Algebras and Quantum Statistical Mechanics 1: C*- and W*-Algebras. Symmetry Groups. Decomposition of States*. Operator Algebras and Quantum Statistical Mechanics. Springer, Berlin, 2003.

[22] M. Unser. Sampling - 50 years after shannon. *Proceedings of the IEEE*, 88(4):569–587, 2000.